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Transportation distance between the Lévy measures and stochastic equations for Lévy-type processes

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Abstract The notion of the transportation distance on the set of the Lévy measures on $\mathbb{R}$ is introduced. A Lévy-type process with a given symbol (state dependent analogue of the characteristic triplet) is proved to be well defined as a strong solution to a stochastic differential equation (SDE) under the assumption of Lipschitz continuity of the Lévy kernel in the symbol w.r.t. the state space variable in the transportation distance. As examples, we construct Gamma-type process and α-stable like process as strong solutions to SDEs.

Keywords Lévy-type processes, existence and uniqueness of the solution to SDE, Gamma-type process, α-stable like process
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1 Introduction

Recently a wide variety of models were proposed in physics, chemistry, biology and econometrics, where the stochastic fluctuations are distributed according to the Lévy law, instead of the more traditional Gaussian one (see for example [1] and the list of references therein). If the parameters in such models are state dependent, then one

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should deal with a *Lévy-type process*. A typical example here is model with so-called Lévy flights, where the correspondent Lévy measure is alpha-stable

$$\Pi(du) = \frac{\lambda du}{|u|^{\alpha+1}}.$$

The particular case corresponds to the model with porous environment [2], and in this case the parameter $\alpha = \alpha(x)$ is state dependent. By definition, the Lévy-type process is such a Markov process that its generator on the functions of the class $\mathcal{C}^2_\infty = \{f : f \in C^2, f(x) \to 0, f'(x) \to 0, \text{ as } |x| \to \infty\}$ takes the form

$$\begin{aligned} Af(x) &= a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\ &\quad + \int_{\mathbb{R}} \big(f(x+y) - f(x) - f'(x)y\mathbf{1}_{|y|\leqslant 1}\big)\Pi(x,dy) \\ &\quad + \int_{\mathbb{R}} \big(f(x+y) - f(x)\big)\mathbf{1}_{|y|>1}\Pi(x,dy), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

Here functions $a : \mathbb{R} \to \mathbb{R}$, $\sigma^2 : \mathbb{R} \to [0;\infty)$ and a *Lévy kernel* $\Pi(x,dy)$ (a measurable function w.r.t. x and a Lévy measure for any $x \in \mathbb{R}$) are the state dependent analogue of the *characteristic triplet* of a Lévy process. Lévy-type processes also may appear as limiting distributions in theorems of Skorokhod's invariance principle-type with the convergence of step-wise processes constructed via Markov chains (see [8, 7]). One of the most natural ways to construct and characterise such processes is to use an SDE approach, e.g. to define the required process as a (strong) solution to a proper SDE. Naturally, one could expect that the uniqueness and existence of solution to such SDE remain valid under Lipschitz continuity type assumptions on the members of the (state-dependent) characteristic triplet, which leads us to the question: what should be the proper form for the Lipschitz continuity condition for the Lévy kernel? To answer this question we consider a *transportation distance* on the set of Lévy measures on $\mathbb{R}$.

The main idea of the transportation distance construction is to present any Lévy measure Π *uniquely* as a transformation of a fixed infinite Lévy measure Π_0 by means of a function c from some prescribed family of functions, and then compare the correspondent functions for two measures. Below we show that for a given Π the respective function c is unique in the properly chosen class and the transportation distance is a metrics on the set of Lévy measures on $\mathbb{R}$. The main gain of this method is that we can obtain any Poisson point measure, with a Lévy kernel as a Lévy measure by means of such function c, from the measure ν_0 with the Lévy measure Π_0. This technique allows one to write a Lévy-type process with a given state dependent characteristic triplet as a solution to an SDE with ν_0 as a Poisson random noise. The solvability of such SDEs and the solution properties are the questions under consideration in this paper.

The idea to present any Lévy kernel as an image of some fixed measure goes back to the works of Skorokhod [9] and Strook [10]. The authors therein treat the equation with indirectly defined function c. The conditions for uniqueness and existence of the solution to such SDE are quite implicit. It appears that in our approach the explicit transportation distance construction allows one to give the conditions in more transparent form.

This paper is organised as follows: first we introduce the notion of the transportation distance and formulate the theorem about the uniqueness and existence of the strong solution to an SDE for Lévy type process. As examples we construct Gamma-type and α-stable like processes as strong solutions to SDEs.

2 Main results

2.1 Transportation distance

Recall that a *Lévy measure* on $\mathbb{R}$ is a measure Π on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} (u^2 \wedge 1)\Pi(du) < \infty.$$

It is convenient for our further purposes to introduce two slightly unusual conventions. First, we admit that a Lévy measure assigns a non-trivial mass to the point 0; that is, it contains a term $m\delta_0(du)$. This term can be even infinite; that is $m \in [0;\infty]$ in general. Second, for two such measures $\Pi_1,\ \Pi_2$ we write

$$\Pi_1 \doteq \Pi_2$$

if they coincide up to the term $m\delta_0(du)$. In other words, $\Pi_1 \doteq \Pi_2$ iff

$$\int_{\mathbb{R}} g(u)\Pi_1(du) = \int_{\mathbb{R}} g(u)\Pi_2(du),$$

for any $g : \mathbb{R} \to \mathbb{R}$ such that $|g(u)| \leqslant u^2 \wedge 1,\ u \in \mathbb{R}$.

Proposition 1. *Let us fix a Lévy measure*

$$\Pi_0(du) = \frac{du}{u^2}, \quad u \in \mathbb{R}$$

and let Π be an arbitrary Lévy measure. Then there exists a unique function $c : \mathbb{R} \to \mathbb{R}$, such that $\Pi = \Pi_0 \circ c^{-1}$ and c satisfies the additional assumptions

1. $c : (-\infty;0] \to (-\infty;0]$ *and* $c : [0;+\infty) \to [0;+\infty)$*;*
2. *c is nondecreasing;*
3. *c is left continuous on* $(-\infty;0]$ *and right continuous on* $[0;+\infty)$.

Example 1. Let $\Pi(du) = \frac{du}{u^2}\mathbf{1}_{u>0}$, then the function c equals $c(v) = v\mathbf{1}_{v>0}$. Note that this function maps all the mass from $[-\infty;0]$ to the point 0, consequently we get a measure $\tilde{\Pi}(du) = \frac{du}{u^2}\mathbf{1}_{u>0} + \infty \cdot \delta_0$ as an image of Π_0 under this c. This is exactly the reason for us to introduce the conventions we've discussed above.

For any $x, y \in \mathbb{R}$ denote

$$\rho(x,y) = |x-y| \wedge 1.$$

Definition 1. For two Lévy measures $\Pi_1,\ \Pi_2$ on $\mathbb{R}$ we put

$$T(\Pi_1,\Pi_2) = \sqrt{\int_{\mathbb{R}} \rho^2(c_{\Pi_1}(u), c_{\Pi_2}(u))\Pi_0(du)}. \tag{2}$$

Proposition 2. *The function T defined above is a metric on the set of the Lévy measures on $\mathbb{R}$.*

Taking into account the proposition above, we call the function T as a *transportation distance*. Proofs of Propositions 1 and 2 are of a technical nature, and we postpone them to Appendix A in order not to overload the text.

2.2 An SDE for a Lévy-type process

Let us consider a state dependent characteristic triplet $(a(x), \sigma^2(x), \Pi(x,\cdot))$ and the function $c(x,v)$ such that $\Pi(x,du) = \Pi_0(\{v : c(x,v) \in du\})$ and for every fixed x the function $c(x,\cdot)$ has the properties listed in Proposition 1. In the same spirit with §1, Chap. 4 [4], one can write the following SDE, whose solution, if exists, is a natural candidate to be a Levy-type process with the given state dependent characteristic triplet:

$$\begin{aligned} dX(t) = a(X(t))dt + \sigma(X(t))dW(t) \\ + \int_{\mathbb{R}} c(X(t-),v)\mathbf{1}_{|c(X(t-),v)|\leqslant 1}\tilde{\nu}_0(dt,dv) \\ + \int_{\mathbb{R}} c(X(t-),v)\mathbf{1}_{|c(X(t-),v)|>1}\nu_0(dt,dv), \end{aligned} \tag{3}$$

where $X(0) = x_0,\ x_0 \in \mathbb{R},\ \nu_0$ is a Poisson point measure with the intensity measure Π_0, W is independent of ν_0 Wiener process and $\tilde{\nu}_0(dt,dv) = \nu_0(dt,dv) - \Pi_0(dv)dt$. However, a thorough look at some particular classes of processes (see Section 3.2 below) shows that it will be more flexible to consider more general SDE

$$\begin{aligned} dX(t) = a(X(t))dt + \sigma(X(t))dW(t) \\ + \sum_{k=1}^{n}\Big[\int_{\mathbb{R}} c_{\Pi_k}(X(t-),v)\mathbf{1}_{|c_{\Pi_k}(X(t-),v)|\leqslant 1}\tilde{\nu}_k(dt,dv) \\ + \int_{\mathbb{R}} c_{\Pi_k}(X(t-),v)\mathbf{1}_{|c_{\Pi_k}(X(t-),v)|>1}\nu_k(dt,dv)\Big], \end{aligned} \tag{4}$$

where

$$\Pi(x,\cdot) = \sum_{k=1}^{n}\Pi_k(x,\cdot) \tag{5}$$

is some decomposition of the given Lévy kernel, $c_{\Pi_k}(x,\cdot)$ are corresponding functions transporting Π_0 to $\Pi_k(x,\cdot)$, $\nu_1,\dots,\nu_n$ are independent copies of the Poisson point measure ν_0, and $\tilde{\nu}_1,\dots,\tilde{\nu}_n$ are corresponding compensated Poisson point measures.

Denote

$$\tilde{a}(x) = a(x) + \Pi(x,\{|u|>1\}).$$

Theorem 1. *Assume that, for some decomposition (5) and some $l \leqslant n$, the following conditions hold true.*

1. *The functions $\tilde{a}$ and σ satisfy the Lipschitz condition w.r.t. ρ, i.e. there exists a constant $L_1 > 0$ such that*

$$|\tilde{a}(x_1) - \tilde{a}(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leqslant L_1\rho(x_1,x_2).$$

2. *There exists a constant $L_2 > 0$ such that*

$$T(\Pi_k(x_1,\cdot),\Pi_k(x_2,\cdot)) \leqslant L_2\rho(x_1,x_2),\quad k=1,\dots,l,\ x_1,x_2\in\mathbb{R}.$$

3. *For* $k = 1, \dots, l$,

$$\sup_{x\in\mathbb{R}} \Pi_k(x, \{u : |u| > 1\}) < \infty.$$

4. *For* $k = l+1, \dots, n$,

$$\sup_{x\in\mathbb{R}} \Pi_k(x, \mathbb{R}) < \infty.$$

Then there exists a unique strong solution to the equation (4). This solution is a Markov process, and its generator on the class C^2_∞ *takes the form (1).*

Proof of Theorem 1: uniqueness. In order not to overload the notation we prove only the case where $n = l = 1$; that is, in fact we shall deal with equation (3). This will not restrict generality, because (a) dealing with l Lipschitz terms instead of one can be made literally in the same way; (b) it is a standard observation that adding the terms with bounded "total intensity" of jumps does not spoil an existence and uniqueness result because we can separate the time instants where the jumps with the bounded total intensity occur, and resolve consequently the SDE on the intervals between these time instants.

Consider two solutions of the equation (3) with the same starting point:

$$\begin{aligned} dX_i(t) &= a(X_i(t))dt + \sigma(X_i(t))dW(t) \\ &\quad + \int_{\mathbb{R}} c(X_i(t-),v)\mathbf{1}_{|c(X_i(t-),v)|\leqslant 1}\tilde{\nu}_0(dt,dv) \\ &\quad + \int_{\mathbb{R}} c(X_i(t-),v)\mathbf{1}_{|c(X_i(t-),v)|>1}\nu_0(dt,dv), \quad i = 1,2. \end{aligned} \tag{6}$$

The standard argument here, say, for SDE's with square integrable noise, would be to use the Itô formula and the Gronwall lemma to prove that $\mathbf{E}(X_1(t) - X_2(t))^2 = 0$. Now because of possible lack of square integrability we shall modify this argument. Namely, we shall prove that for every $t \in [0,1]$

$$\mathbf{E}\big((X_1(t) - X_2(t))^2 \wedge 1\big) = 0. \tag{7}$$

To do this, we apply the smoothing cut-off technique developed in [3]. Namely we shall apply the Itô formula to

$$F(y) = \arctan y^2,$$

see Chap. 2, [5]. To use the Itô formula we need an auxiliary construction. For some $\varepsilon > 0$ we can rewrite the process $Y(t) = X_1(t) - X_2(t)$ as follows

$$\begin{aligned} Y(t) &= \int_0^t (\tilde{a}(X_1(s)) - \tilde{a}(X_2(s)))ds + \int_0^t (\sigma(X_1(s)) - \sigma(X_2(s)))dW(s) \\ &\quad + \int_0^t \int_{|v|\leqslant\varepsilon} (c(X_1(s-),v) - c(X_2(s-),v))\tilde{\nu}_0(dv,ds) \\ &\quad + \int_0^t \int_{|v|>\varepsilon} (c(X_1(s-),v) - c(X_2(s-),v))\nu_0(dv,ds) \\ &\quad - \int_0^t \int_{|v|>\varepsilon} [\tau(c(X_1(s),v)) - \tau(c(X_2(s),v))]\Pi_0(dv)ds \\ &\quad + \int_0^t \int_{|v|\leqslant\varepsilon} [c(X_1(s),v) - \tau(c(X_1(s),v))]\Pi_0(dv)ds \\ &\quad - \int_0^t \int_{|v|\leqslant\varepsilon} [c(X_2(s),v) - \tau(c(X_2(s),v))]\Pi_0(dv)ds, \end{aligned} \tag{8}$$

where

$$\tau(v) = (|v| \wedge 1)\,\mathrm{sign}(v), \quad x, v \in \mathbb{R}.$$

Using the Itô formula we get

$$\begin{aligned}
F(Y(t)) = &\int_0^t (\tilde{a}(X_1(s)) - \tilde{a}(X_2(s)))F'(Y(s))ds \\
&+ \int_0^t (\sigma(X_1(s)) - \sigma(X_2(s)))F'(Y(s))dW(s) \\
&+ \int_0^t \frac{1}{2}(\sigma(X_1(s)) - \sigma(X_2(s)))^2 F''(Y(s))ds \\
&+ \int_0^t \int_{|v|\leqslant\varepsilon} [F(Y(s-) + (c(X_1(s-), v) - c(X_2(s-), v))) \\
&- F(Y(s-))]\tilde{\nu}_0(dv, ds) \\
&+ \int_0^t \int_{|v|\leqslant\varepsilon} [F(Y(s) + (c(X_1(s), v) - c(X_2(s), v))) \\
&- (c(X_1(s), v) - c(X_2(s), v))F'(Y(s)) - F(Y(s))]\Pi_0(dv)ds \\
&+ \int_0^t \int_{|v|>\varepsilon} [F(Y(s-) + (c(X_1(s-), v) - c(X_2(s-), v))) \\
&- F(Y(s-))]\nu_0(dv, ds) \\
&- \int_0^t \int_{|v|>\varepsilon} [\tau(c(X_1(s), v)) - \tau(c(X_2(s), v))]F'(Y(s))\Pi_0(dv)ds \\
&+ \int_0^t \int_{|v|\leqslant\varepsilon} [c(X_1(s), v) - \tau(c(X_1(s), v))]F'(Y(s))\Pi_0(dv)ds \\
&- \int_0^t \int_{|v|\leqslant\varepsilon} [c(X_2(s), v) - \tau(c(X_2(s), v))]F'(Y(s))\Pi_0(dv)ds. \qquad (9)
\end{aligned}$$

After rearrangements, we get finally the formula

$$F(Y(t)) = M_t + \int_0^t g(Y(s), X_1(s), X_2(s))\Pi_0(dv)ds, \qquad (10)$$

where

$$\begin{aligned}
M_t = &\int_0^t (\sigma(X_1(s)) - \sigma(X_2(s)))F'(Y(s))dW(s) \\
&+ \int_0^t \int_{\mathbb{R}} [F(Y(s-) + (c(X_1(s-), v) - c(X_2(s-), v))) - F(Y(s-))]\tilde{\nu}_0(dv, ds)
\end{aligned} \qquad (11)$$

is a martingale, and

$$\begin{aligned}
g(y, x_1, x_2) = &(\tilde{a}(x_1) - \tilde{a}(x_2))F'(y) \\
&+ \frac{1}{2}(\sigma(x_1) - \sigma(x_2))^2 F''(y) \\
&+ \int_0^t \int_{\mathbb{R}} [F(y + (c(x_1, v) - c(x_2, v))) \\
&- [\tau(c(x_1, v)) - \tau(c(x_2, v))]F'(y) - F(y)]\Pi_0(dv)ds \\
= &g_1(y, x_1, x_2) + g_2(y, x_1, x_2) + g_3(y, x_1, x_2). \qquad (12)
\end{aligned}$$

Observe that

$$y \wedge 1 \leqslant \frac{4}{\pi} \arctan y, \quad y \geqslant 0. \tag{13}$$

In addition, for the function F and its derivatives, we have the following explicit expressions and bounds:

$$F'(y) = \frac{2y}{1+y^4}, \qquad F''(y) = 2\frac{1-y^4}{(1+y^4)^2}, \tag{14}$$

$$|F''(y)| \leqslant 2, \qquad |F'(y)| \leqslant \frac{1+y^2}{1+y^4} \leqslant 2, \tag{15}$$

$$|F'(y)||y| = \frac{2y^2}{1+y^4} \leqslant (2y^2) \wedge 1 \leqslant \frac{4}{\pi} F(y), \tag{16}$$

$$|F(y+\delta) - F(y) - F'(y)\delta| \leqslant \frac{\delta^2}{2} \sup_v |F''(v)| \leqslant \delta^2. \tag{17}$$

Using that, we bound every term in the r.h.s. of (12) for any triple (y, x_1, x_2) which satisfy $y = x_1 - x_2$. Observe that $Y(s) = X_1(s) - X_2(s)$, hence to estimate the right hand side of (10) we can restrict our consideration to this class of triples (y, x_1, x_2).

Estimate (16) and condition 1 yield

$$g_1(y, x_1, x_2) \leqslant \frac{4L_1}{\pi} F(y), \quad y = x_1 - x_2.$$

Estimates (15), (13) and condition 1 yield

$$g_2(y, x_1, x_2) \leqslant \frac{2L_1^2}{\pi} F(y), \quad y = x_1 - x_2.$$

To estimate $g_3(y, x_1, x_2)$, we re-write it in the following way

$$g_3(y, x_1, x_2) = \int_{\{v:|c(x_1,v)-c(x_2,v)\}|\leqslant 1} + \int_{\{v:|c(x_1,v)-c(x_2,v)|>1\}} [f(y, x_1, x_2)] \varPi_0(dv),$$

where

$$f(y, x_1, x_2) := F\big(y + (c(x_1,v) - c(x_2,v))\big) - F(y) - F'(y)\big(\tau(c(x_1,v)) - \tau(c(x_2,v))\big)$$

and note that in any case the absolute value of the $f(y, x_1, x_2)$ does not exceed $\pi + 4$. In the case when $|c(x_1,v) - c(x_2,v)| \leqslant 1$, we have the inequality

$$F\big(y + (c(x_1,v) - c(x_2,v))\big) - F(y) \leqslant F'(y)\big(c(x_1,v) - c(x_2,v)\big) + \big(c(x_1,v) - c(x_2,v)\big)^2,$$

which comes from (17). Hence, after simple re-arrangements, we get

$$\begin{aligned} g_3(y, x_1, x_2) \leqslant{} & (\pi + 4) \int_{\mathbb{R}} \big((c(x_1,v) - c(x_2,v))^2 \wedge 1\big) \varPi_0(dv) \\ & + F'(y) \int_{|c(x_1,v)-c(x_2,v)|\leqslant 1} \big[(c(x_1,v) - c(x_2,v)) \\ & - \big(\tau(c(x_1,v)) - \tau(c(x_2,v))\big)\big] \varPi_0(dv) \end{aligned}$$

$$
\begin{aligned}
&= (\pi + 4)T^2(\Pi(x_1,\cdot), \Pi(x_2,\cdot)) \\
&\quad + F'(y)\int_{|c(x_1,v)-c(x_2,v)|\leqslant 1}\big[(c(x_1,v)-c(x_2,v)) \\
&\quad - (\tau(c(x_1,v)) - \tau(c(x_2,v)))\big]\Pi_0(dv).
\end{aligned}
$$

Next, the function τ is Lipschitz with the constant 1, hence the absolute value of the inner function in the last integral is bounded by

$$
|c(x_1,v)-c(x_2,v)| + |\tau(c(x_1,v)) - \tau(c(x_2,v))| \leqslant 2(|c(x_1,v)-c(x_2,v)| \wedge 1),
$$

since the domain of integration is $\{v : |c(x_1,v)-c(x_2,v)| \leqslant 1\}$. If, within this domain, we have $|c(x_1,v)| \leqslant 1$, $|c(x_2,v)| \leqslant 1$, then $\tau(c(x_i,v)) = c(x_i,v)$, $i = 1,2$ and the inner function in the last integral equals zero. Hence by the Cauchy inequality

$$
\begin{aligned}
&\int_{|c(x_1,v)-c(x_2,v)|\leqslant 1}\big[\tau(c(x_1,v)-c(x_2,v)) - (\tau(c(x_1,v)) - \tau(c(x_2,v)))\big]\Pi_0(dv) \\
&\quad \leqslant 2\int_{|c(x_1,v)|>1 \text{ or } |c(x_2,v)|>1}(|c(x_1,v)-c(x_2,v)| \wedge 1)\Pi_0(dv) \\
&\quad \leqslant 2\Big(\int_{\mathbb{R}^2}((c(x_1,v)-c(x_2,v))^2 \wedge 1)\Pi_0(dv)\Big)^{1/2} \\
&\qquad \times (\Pi_0(\{v : |c(x_1,v)| > 1 \text{ or } |c(x_2,v)| > 1\}))^{1/2}.
\end{aligned}
$$

Denote

$$
B = \sup_{x\in\mathbb{R}} \Pi(x, \{u : |u| > 1\});
$$

see condition 3. Then, using the elementary inequality $2ab \leqslant a^2 + b^2$ we obtain

$$
g_3(y,x_1,x_2) \leqslant (F'(y))^2 + (\pi + 4 + 2B)T^2(\Pi(x_1,\cdot), \Pi(x_2,\cdot)). \tag{18}
$$

Recall that $y = x_1 - x_2$, hence by the Lipschitz condition on $\Pi(x,\cdot)$ we have

$$
T^2(\Pi(x_1,\cdot), \Pi(x_2,\cdot)) \leqslant L_2^2(|x_1 - x_2|^2 \wedge 1) \leqslant \frac{4L_2^2}{\pi}F(y).
$$

Next, combining the first inequality in (14) and the second one in (15), we get

$$
|F'(y)| \leqslant 2(|y| \wedge 1).
$$

Hence, using (13) once again, we can write

$$
(F'(y))^2 \leqslant 4(y^2 \wedge 1) \leqslant \frac{16}{\pi}F(y), \quad y \in \mathbb{R}. \tag{19}
$$

This gives finally

$$
g_3(y,x_1,x_2) \leqslant \frac{16 + 4L_2^2(\pi + 4 + 2B)}{\pi}F(y). \tag{20}
$$

Summarising all the above we get

$$
g(y,x_1,x_2) \leqslant \frac{4L_1 + 2L_1^2 + 16 + 4L_2^2(\pi + 4 + 2B)}{\pi}F(y). \tag{21}
$$

Putting

$$Q := \frac{4L_1 + 2L_1^2 + 16 + 4L_2^2(\pi + 4 + 2B)}{\pi},$$

we'll get that

$$\mathbf{E}F(Y(t)) \leqslant Q\int_0^t \mathbf{E}F(Y(s))ds \tag{22}$$

since M_t is a martingale and hence $\mathbf{E}M_t = 0$. Thus by the Gronwall lemma and non-negativity of the function F we get

$$\mathbf{E}F(Y(t)) = 0. \tag{23}$$

Consequently by (13) we get (7). This completes the proof of the uniqueness part. □

Proof of Theorem 1: existence. Consider the sequence of the successive approximations for the solution of the equation (3):

$$\begin{aligned} X_0(t) &= x_0, \quad t \in [0;1], \\ X_{n+1}(t) &= x_0 + \int_0^t a(X_n(s))ds + \int_0^t \sigma(X_n(s))dW(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} c(X_n(s-), v)\mathbf{1}_{|c(X_n(s-),v)|\leqslant 1}[\nu_0(ds, dv) - \Pi_0(dv)ds] \\ &\quad + \int_0^t \int_{\mathbb{R}} c(X_n(s-), v)\mathbf{1}_{|c(X_n(s-),v)|>1}\nu_0(ds, dv). \end{aligned} \tag{24}$$

Using the same arguments as for the estimate (22) we can get for the processes $Y_{n+1}(t) = X_{n+1}(t) - X_n(t)$ the following

$$\mathbf{E}FY_{n+1}(t) \leqslant Q\int_0^t \mathbf{E}F(Y_n(s))ds \tag{25}$$

Using the estimates (13) and

$$F(y) \leqslant y^2 \wedge (\pi/2) \leqslant (\pi/2)(y \wedge 1),$$

we obtain

$$\mathbf{E}(Y_{n+1}(t)^2 \wedge 1) \leqslant \tilde{Q}\int_0^t \mathbf{E}(Y_n(s)^2 \wedge 1)ds \leqslant \frac{\tilde{Q}^n}{n!}, \quad t \in [0;1], \tag{26}$$

where $\tilde{Q} = 2Q$. Consequently

$$\sup_{t\in[0;1]} \mathbf{E}\rho^2(X_{n+1}(t), X_n(t)) = 0, \quad n \geqslant 1. \tag{27}$$

Therefore there exists a process X such that it satisfies equation (3) and

$$\lim_{n\to\infty} \sup_{t\in[0;1]} \mathbf{E}\rho^2(X_n(t), X(t)) = 0. \tag{28}$$

Passing to the limit in (24) will yield that X is a solution to (3); this argument is standard and we omit the details.

The proof of the Markov property for X is standard as well, and it is omitted. The formula (1) for its generator follows from the Itô formula in a standard way, and again, we omit the details. □

3 Examples

3.1 The Gamma-type process

Let us consider *Gamma process* with the Lévy measure $\Pi(du) = \frac{\gamma e^{-\lambda u}}{u} du$, $y > 0$, $\gamma, \lambda > 0$. Here parameter γ can be interpreted as a rate of jump arrivals, and λ as an effective size of a jump. The proposition below gives the estimates for the transportation distance between two Gamma measures, with one of the above parameters being fixed and other one varying.

Proposition 3. *1. Let Π_j, $j = 1, 2$ be two Gamma measures with the same parameter λ and different parameters $0 < \gamma_1 < \gamma_2$. Then there exists a constant $D > 0$ such that the following bound holds true*

$$T(\Pi_1, \Pi_2) \leqslant \frac{D}{\lambda^2}(\gamma_2 - \gamma_1)(\log \gamma_2 - \log \gamma_1). \tag{29}$$

2. Let Π_j, $j = 1, 2$ be two Gamma measures with the same parameter γ and different parameters $0 < \lambda_1 < \lambda_2$. Then there exists a constant $\tilde{D} > 0$ such that the following bound holds true

$$T(\Pi_1, \Pi_2) \leqslant \gamma \tilde{D}(\lambda_2 - \lambda_1)^2 \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + 1 \right). \tag{30}$$

For the proof we refer the reader to Appendix B.

We call a *Gamma-type process* a Lévy-type process with a characteristic triplet $(0, 0, \Pi(x, du))$ where a Lévy kernel is $\Pi(x, du) = \frac{\gamma(x) e^{-\lambda(x) u}}{u} du$, $u > 0$. If we assume that only the size of jumps varies, i.e. the parameter γ is fixed, then the corresponding SDE can be easily written in the form:

$$dX_t = \frac{1}{\lambda(X_{t-})} dX_t^1,$$

where X^1 denotes the Gamma process with the same parameter γ and $\lambda(x) = 1$. Meanwhile the case where the intensity of jumps $\gamma = \gamma(x)$ varies is not so easy to treat because, heuristically, one should introduce the change of time into the Lévy-Itô formula.

Nevertheless using Theorem 1 in this case is also manageable. If $\gamma(x)$ is Lipschitz continuous, then all the conditions of Theorem 1 are verified. The condition 2 follows from (30), the condition 3 is easy to verify straightforwardly. Thus Gamma-type process in this case can be written as a solution to equation (3) with $c(x, v) = \varphi(\frac{1}{v\gamma(x)})/\lambda$.

3.2 The α-stable like process

Recall that for an *α-stable process*, its Lévy measure has the form

$$\Pi(du) = \left(\alpha \lambda^- |u|^{-\alpha-1} \mathbf{1}_{(-\infty;0)}(u) + \alpha \lambda^+ u^{-\alpha-1} \mathbf{1}_{(0,\infty)}(u) \right) du, \tag{31}$$

where $\alpha \in (0, 2)$ and $\lambda^-, \lambda^+ \geqslant 0$. Analogously to Proposition 2 in [3] one can obtain the following

Proposition 4. *1. Let Π_j, $j = 1,2$ be two α-stable measures with the same shape parameter α and different scale parameters $\lambda^{\pm}_{1,2} \in [a,b]$, $a,b > 0$. In this case*

$$T(\Pi_1,\Pi_2) \leqslant \left(\frac{2}{2-\alpha}\right)\big(|(\lambda^-_1)^{1/\alpha} - (\lambda^-_2)^{1/\alpha}|^\alpha + |(\lambda^+_1)^{1/\alpha} - (\lambda^+_2)^{1/\alpha}|^\alpha\big). \tag{32}$$

2. *Let Π_j, $j = 1,2$ be two α-stable measures with the same scale parameters λ^+, λ^-, but different shape parameters $\alpha_{1,2} \in [c,d] \subset (0;2)$. In this case there exists a constant $D > 0$ such that the estimate is as follows*

$$T(\Pi_1,\Pi_2) \leqslant D|\alpha_2 - \alpha_1|^{d/2}. \tag{33}$$

We call as an α-stable like process, a Lévy-type process with a characteristic triplet $(0,0,\Pi(x,du))$, where a Lévy kernel is

$$\Pi(x,du) = \alpha(x)\lambda^-(x)|u|^{-\alpha(x)-1}\mathbf{1}_{u<0}\,du + \alpha(x)\lambda^+(x)u^{-\alpha(x)-1}\mathbf{1}_{u>0}\,du. \tag{34}$$

Note that even if the functions α, $\lambda^{\pm}$ are Lipschitz continuous itself, then by estimates (32) and (33) we observe that the kernel $\Pi(x,du)$ is not Lipschitz continuous in transportation distance, hence we can not use Theorem 1 to construct the required Lévy-type process as a strong solution to the SDE (3). Nevertheless, we are still able to use Theorem 1: to do that, we represent the kernel $\Pi(x,du)$ as a sum of the Lipschitz continuous kernel and the one of a bounded total jump intensity.

Consider the *damped α-stable* Lévy measure

$$\Pi(du) = \alpha\lambda^-|u|^{-\alpha-1}\mathbf{1}_{(-1,0)}(u) + \alpha\lambda^+u^{-\alpha-1}\mathbf{1}_{(0,1)}(u)\,du. \tag{35}$$

Proposition 5. *1. Let Π_j, $j = 1,2$ be two damped α-stable measures with the same shape parameter α and different scale parameters $\lambda^{\pm}_{1,2} \in [a,b]$, $a,b > 0$. In this case there exists some constant $D_1 > 0$ such that*

$$T(\Pi_1,\Pi_2) \leqslant D_1(|\lambda^-_1 - \lambda^-_2| + |\lambda^+_1 - \lambda^+_2|). \tag{36}$$

2. *Let Π_j, $j = 1,2$ be two damped α-stable measures with the same scale parameter λ^+, but different shape parameters $\alpha_{1,2} \in [c,d] \subset (0;2)$. In this case there exists some constant $D_2 > 0$ such that*

$$T(\Pi_1,\Pi_2) \leqslant D_2|\alpha_1 - \alpha_2|. \tag{37}$$

The proof is given in Appendix B.

Now we can rewrite the kernel (34) as follows:

$$\begin{aligned}\Pi(x,du) &= \alpha(x)\lambda^-(x)|u|^{-\alpha(x)-1}\mathbf{1}_{u\in(-1,0)}\,du + \alpha(x)\lambda^+(x)u^{-\alpha(x)-1}\mathbf{1}(u)_{(0,1)}\,du\\ &\quad + \alpha(x)\lambda^-(x)|u|^{-\alpha(x)-1}\mathbf{1}_{u\in(-\infty,-1)} + \alpha(x)\lambda^+(x)u^{-\alpha(x)-1}\mathbf{1}(u)_{(1,+\infty)}\,du\\ &= \Pi_1(x,du) + \Pi_2(x,du).\end{aligned} \tag{38}$$

Hence, if the functions $\lambda^-(x)$ and $\lambda^+(x)$ are Lipschitz continuous and bounded, and the function $\alpha(x)$ is Lipschitz continuous and takes its values in a segment $[c,d] \subset$

$(0, 2)$, then the kernel $\Pi(x, du)$ admits a decomposition such that all the conditions of Theorem 1 are verified. Correspondingly, the *α-stable like* process with the Lévy kernel (34) can be obtained as a solution to the SDE

$$dX(t) = \sum_{k=1}^{2} \Big[\int_{\mathbb{R}} c_{\Pi_k}(X(t-), v) \mathbf{1}_{|c_{\Pi_k}(X(t-),v)| \leqslant 1} \tilde{\nu}_k(dt, dv) + \int_{\mathbb{R}} c_{\Pi_k}(X(t-), v) \mathbf{1}_{|c_{\Pi_k}(X(t-),v)| > 1} \nu_k(dt, dv) \Big]. \tag{39}$$

A Proof of the properties of transportation distance

Proof of Proposition 1. The statement of the proposition is equivalent to the fact that there exists an unique nondecreasing function $\kappa_\Pi : \mathbb{R} \to \mathbb{R}$ cáglád on $(-\infty; 0]$ and cádlág on $[0; +\infty)$ such that

$$\begin{aligned} \Pi(\{u : u > x\}) &= \Pi_0(\{v : v > \kappa_\Pi(x)\}), \quad x \geqslant 0, \\ \Pi(\{u : u < -x\}) &= \Pi_0(\{v : v < \kappa_\Pi(-x)\}), \quad x \geqslant 0. \end{aligned} \tag{40}$$

Here and below we understand $\Pi(\{u : u > 0\}) = \lim_{x' \to 0, x' > 0} \Pi(\{u : u > x'\})$. Thus one immediately get the function κ_Π explicitly:

$$\kappa_\Pi(x) := \begin{cases} \frac{1}{\Pi(\{u:u>x\})}, & x \geqslant 0; \\ -\frac{1}{\Pi(\{u:u<x\})}, & x \leqslant 0. \end{cases} \tag{41}$$

When it is needed let us put $1/\infty = 0$. The function κ_Π is non-decreasing, left-continuous on $(-\infty; 0]$ and right-continuous on $[0; +\infty)$ respectively, thus we can take the function c as the generalised inverse of the function κ when $x \geqslant 0$ and when $x \leqslant 0$, that is

$$c_\Pi(y) = \begin{cases} \sup\{x \in (0; +\infty) : \kappa_\Pi(x) < y\}, & y \in [0; +\infty); \\ \inf\{x \in (-\infty; 0) : \kappa_\Pi(x) > y\}, & y \in (-\infty; 0]. \end{cases} \tag{42}$$

Now we should check that for the function c defined in (42)

$$\begin{aligned} \{\varepsilon \geqslant 0 : c_\Pi(\varepsilon) > x\} &= (\kappa_\Pi(x); +\infty) \quad x \geqslant 0 \quad \text{and} \\ \{\varepsilon < 0 : c_\Pi(\varepsilon) < -x\} &= (-\infty; \kappa_\Pi(-x)), \quad x \leqslant 0. \end{aligned} \tag{43}$$

Indeed, let us consider for $x \geqslant 0$

$$\begin{aligned} \{\varepsilon \geqslant 0 : c_\Pi(\varepsilon) > x\} &= \Big\{\varepsilon \geqslant 0 : \sup_{y \in [0; +\infty)} (\kappa_\Pi(y) < \varepsilon) > x\Big\} = \bigcup_{y > x} \{\kappa_\Pi(y) < \varepsilon\} \\ &= \bigcup_{y > x} (\kappa_\Pi(y); +\infty) = (\kappa_\Pi(x); +\infty). \end{aligned}$$

The second part of (43) can be obtained using the same reasoning. □

Proof of Proposition 2. To verify the statement of the proposition we need to check three properties of the metric. First note that the quantity T is nonnegative because ρ is a metric. If $T(\Pi_1, \Pi_2) = 0$ then $\rho(c_{\Pi_1}(u), c_{\Pi_2}(u)) = 0,\ u \in \mathbb{R}$. It implies that

$$c_{\Pi_1}(u) = c_{\Pi_2}(u), \quad u \in \mathbb{R},$$

thus

$$\Pi_1 = \Pi_2$$

because c_Π uniquely identifies the measure Π. The symmetry axiom for the metric T follows from the one for ρ. The triangle inequality follows from the triangle inequality for the metric ρ and the Minkovsky inequality:

$$\begin{aligned} T(\Pi_1, \Pi_2) &\leqslant \left(\int_{\mathbb{R}} (\rho(c_{\Pi_1}(u), c_{\Pi_3}(u)) + \rho(c_{\Pi_3}(u), c_{\Pi_2}(u))^2)\, \Pi_0(du) \right)^{1/2} \\ &\leqslant T(\Pi_1, \Pi_3) + T(\Pi_3, \Pi_2). \end{aligned}$$ □

B Explicit calculations for transportation distance

Proof of Proposition 3: statement 1. First let us denote the function $\psi_\gamma(x) = \int_x^{+\infty} \frac{\gamma e^{-u}}{u} du$. Note that function $\psi_1(x) = -\mathbf{Ei}(-x),\ x > 0$, where $\mathbf{Ei}(x)$ is the well-known exponential integral (see for instance Chap. VI [6]). Also let us denote the inverse to $\psi_\gamma(x)$ as $\varphi_\gamma(y),\ y > 0$. It can be easily seen that

$$c_{\Pi_j}(y) = \frac{1}{\lambda} \varphi_{\gamma_j}\left(\frac{1}{y}\right), \quad y > 0,\ j = 1, 2.$$

Using the fact that $|x - y| \geqslant \rho(x, y)$ we get the following

$$T(\Pi_1, \Pi_2) \leqslant \int_0^{+\infty} (c_{\Pi_1}(y) - c_{\Pi_2}(y))^2 \frac{dy}{y^2}.$$

Making a change of variables in the integral in the r.h.s. of the last inequality we get

$$\int_0^{+\infty} (c_{\gamma_1}(y) - c_{\gamma_2}(y))^2 \frac{dy}{y^2} = \frac{1}{\lambda^2} \int_0^{\infty} (\varphi_{\gamma_1}(v) - \varphi_{\gamma_2}(v))^2 dv.$$

Consequently

$$T(\Pi_1, \Pi_2) \leqslant \frac{1}{\lambda^2} \int_0^{\infty} (\varphi_{\gamma_1}(v) - \varphi_{\gamma_2}(v))^2 dv. \tag{44}$$

To bound the expression in the r.h.s. of (44) we need some properties of the function ψ_γ. First note that $\psi_\gamma(x) = \gamma \psi_1(x)$ for any $\gamma > 0$ and $x > 0$. Consequently $\varphi_\gamma(v) = \varphi_1(v/\gamma)$. We have that $\partial_\gamma \psi_\gamma(x) = \psi_1(x)$ and $\psi_\gamma(\varphi_\gamma(u)) = u$. Differentiating both sides of the latter equation we immediately obtain $\partial_\gamma \varphi_\gamma(u) = \frac{1}{\gamma^2} \varphi_\gamma(u) e^{\varphi_\gamma(u)} u$. Now the integral in the r.h.s. of (44) equals $\int_0^\infty (\int_{\gamma_1}^{\gamma_2} \partial_\gamma \varphi_\gamma(v) d\gamma)^2 dv$. By the Cauchy-Schwarz inequality we get that the last integral is not greater then

$$(\gamma_1 - \gamma_2) \int_0^{\infty} \int_{\gamma_1}^{\gamma_2} \frac{1}{\gamma^4} \varphi_\gamma^2(v) e^{2\varphi_\gamma(v)} v^2 d\gamma dv.$$

Changing the order of integration we have

$$(\gamma_1 - \gamma_2) \int_{\gamma_1}^{\gamma_2} \frac{1}{\gamma^4} \int_0^\infty \varphi_\gamma^2(v) e^{2\varphi_\gamma(v)} v^2 dv d\gamma.$$

Using that $\varphi_1(v/\gamma) = \varphi_\gamma(v)$ and changing variables in the inner integral we obtain

$$(\gamma_1 - \gamma_2) \int_{\gamma_1}^{\gamma_2} \frac{1}{\gamma} \int_0^\infty \varphi_1^2(u) e^{2\varphi_1(u)} u^2 du d\gamma. \tag{45}$$

Using the well known asymptotics of the function ψ_1 (see Chap. VI [6]) we get the asymptotical behaviour of the function φ_1: there exist constants $B_1, B_2 > 0$ and $0 < b_1 < 1$, $b_2 > 1$ such that $\varphi_1(u) \leqslant B_1 \log \frac{1}{Cu}$, $u \in (0; b_1)$ and $\varphi_1(u) \leqslant B_2 e^{-u}$, $u \in [b_2; +\infty)$, here C is the Euler-Mascheroni constant. Using this we obtain

$$\int_0^\infty \varphi_1^2(u) e^{2\varphi_1(u)} u^2 du \leqslant \frac{B_1}{C^2} \int_0^{b_1} (\log u)^2 du + \int_{b_1}^{b_2} \varphi_1^2(u) e^{2\varphi_1(u)} u^2 du + B_2 \int_{b_2}^\infty e^{-2u} e^{2e^{-u}} u^2 du.$$

Every integral in the r.h.s. of the last expression is finite, thus there exists a constant $D > 0$ such that the integral at (45) is bounded from above by $(\gamma_1 - \gamma_2) \int_{\gamma_1}^{\gamma_2} \frac{D}{\gamma} d\gamma$. Thus we obtain (29). □

Proof of Proposition 3: statement 2. Now we consider the function

$$\psi^\lambda(x) = \int_x^{+\infty} \frac{e^{-\lambda u}}{u} du = \int_{\lambda x}^{+\infty} \frac{e^{-u}}{u} du.$$

Note that $\psi^\lambda(x) = \psi_1(\lambda x)$, where $\psi_1(x) = -\mathbf{Ei}(-x)$, $x > 0$. Thus the inverse function $\varphi^\lambda(y) = \varphi_1(y)/\lambda$, where φ_1 is the inverse to ψ_1. In this case $c_{\Pi_j}(y) = \varphi^{\lambda_j}(\frac{1}{\gamma y})$, $0 \leqslant y \leqslant 1$, $j = 1, 2$. As in the proof of Proposition 3, statement 1 we reduce the calculations to the estimation of the integral

$$\int_0^{+\infty} (c_{\Pi_1}(y) - c_{\Pi_2}(y))^2 \frac{dy}{y^2}.$$

Analogously to (44) we obtain

$$T(\Pi_1, \Pi_2) \leqslant \gamma \int_0^\infty (\varphi^{\lambda_1}(v) - \varphi^{\lambda_2}(v))^2 dv. \tag{46}$$

Using the same argument as for the $\partial_\gamma \varphi_\gamma$ we get that $\partial_\lambda \varphi^\lambda(u) = -\frac{\varphi^\lambda u}{\lambda}$. Making the same steps as for (45) we obtain

$$(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} \frac{1}{\lambda^4} \int_0^\infty \varphi_1^2(u) du d\lambda. \tag{47}$$

Using the asymptotics of φ_1 we conclude that there exists a constant $\tilde{D} > 0$ such that the integral in (47) is not greater than

$$\tilde{D}(\lambda_2 - \lambda_1)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right). \tag{48}$$

Using this we get (30). □

Proof of Proposition 4: statement 1. It can be straightforwardly verified that

$$c_{\Pi_j}(y) = -\left(1 - \frac{1}{\lambda_j^- y}\right)^{-1/\alpha} \mathbf{1}_{y<0} + \left(1 + \frac{1}{\lambda_j^+ y}\right)^{-1/\alpha} \mathbf{1}_{y>0}, \quad j = 1, 2. \tag{49}$$

Using an obvious estimate $\rho(x, y) \leqslant |x - y|$ we get

$$T(\Pi_1, \Pi_2) \leqslant \int_{-\infty}^{0} \left(\left(1 - \frac{1}{\lambda_1^- y}\right)^{-1/\alpha} - \left(1 - \frac{1}{\lambda_2^- y}\right)^{-1/\alpha} \right)^2 \frac{dy}{y^2} + \int_{0}^{+\infty} \left(\left(1 + \frac{1}{\lambda_1^+ y}\right)^{-1/\alpha} - \left(1 + \frac{1}{\lambda_2^+ y}\right)^{-1/\alpha} \right)^2 \frac{dy}{y^2}. \tag{50}$$

Making a change of variables in the first integral we get that the r.h.s. of the last expression is not greater than

$$T(\Pi_1, \Pi_2) \leqslant \int_{0}^{+\infty} \left(\left(1 + \frac{1}{\lambda_1^- y}\right)^{-1/\alpha} - \left(1 + \frac{1}{\lambda_2^- y}\right)^{-1/\alpha} \right)^2 \frac{dy}{y^2} + \int_{0}^{+\infty} \left(\left(1 + \frac{1}{\lambda_1^+ y}\right)^{-1/\alpha} - \left(1 + \frac{1}{\lambda_2^+ y}\right)^{-1/\alpha} \right)^2 \frac{dy}{y^2}. \tag{51}$$

Note that we can estimate only one of these integrals. To do this we consider

$$\frac{\partial}{\partial \lambda^+} \left(1 + \frac{1}{\lambda^+ y}\right)^{-1/\alpha} = -\frac{1}{y(\lambda^+)^2} (-1/\alpha) \left(1 + \frac{1}{\lambda^+ y}\right)^{-1/\alpha - 1}. \tag{52}$$

Rewriting the r.h.s. of the last inequality we get

$$\frac{1}{\lambda^+} \frac{\frac{1}{\alpha \lambda^+ y}}{1 + \frac{1}{\lambda^+ y}} \left(1 + \frac{1}{\lambda^+ y}\right)^{-1/\alpha} \leqslant \frac{1}{\alpha \lambda^+}, \quad y > 0. \tag{53}$$

Using this and Cauchy-Schwarz inequality we obtain

$$\int_{0}^{+\infty} \left(\left(1 + \frac{1}{\lambda_1^+ y}\right)^{-1/\alpha} - \left(1 + \frac{1}{\lambda_2^+ y}\right)^{-1/\alpha} \right)^2 \frac{dy}{y^2} \leqslant (\lambda_1^+ - \lambda_2^+) \int_{0}^{+\infty} \int_{\lambda_1^+}^{\lambda_2^+} \frac{1}{(\alpha \lambda y)^2} d\lambda dy. \tag{54}$$

Thus we get the estimate (36). □

Proof of Proposition 4: statement 2. It is now clear that the estimate (37) can be obtained after estimation of the following integral

$$\int_{0}^{+\infty} \left(\left(1 + \frac{1}{\lambda^+ y}\right)^{-1/\alpha_1} - \left(1 + \frac{1}{\lambda^+ y}\right)^{-1/\alpha_2} \right)^2 \frac{dy}{y^2}, \tag{55}$$

or after changing variables

$$\lambda^+ \int_{1}^{\infty} (v^{-1/\alpha_1} - v^{-1/\alpha_2})^2 dv. \tag{56}$$

The r.h.s. of the last equation equals $\lambda^+ \int_1^\infty \int_{\alpha_1}^{\alpha_2} (\partial/\partial \alpha v^{-1/\alpha})^2 d\alpha dv$. By Cauchy-Schwarz inequality we get that this is not greater then

$$\lambda^+(\alpha_1 - \alpha_2) \int_1^\infty \int_{\alpha_1}^{\alpha_2} \frac{1}{\alpha^4} v^{-2/\alpha} (\log v)^2 dv d\alpha.$$

As the integrand in the inner integral is integrable we get estimate (37). □

References

[1] Barndorff-Nielsen, O.E., et al.: Lévy Processes: Theory and Applications. Birkhäuser, Boston (2001). MR1833689

[2] Bouchaud, J., Georges, A.: Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications, Phys. Rep. **195**(4/5), 127–203 (1990). MR1081295

[3] Gairing, J., Högele, M., Kosenkova, T., Kulik, A.: Coupling distances between Lévy measures and applications to noise sensitivity of SDE. Stoch. and Dyn. (2014). doi: 10.1142/S0219493715500094

[4] Gikhman, I.I., Skorokhod, A.V.: Stochastic Differential Equations and Their Applications. Naukova Dumka, Kyiv (1982) (in Russian). MR0678374

[5] Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland, Kodansha Ltd., Tokyo (1981). MR0637061

[6] Janke, E., Emde, F., Lösch, F.: Tafeln Hoeherer Funktionen. B.G. Teubner Verlagsgesellschaft, Stuttgart (1966)

[7] Koroliuk, V.S., Limnios, N., Samoilenko, I.V.: Lévy approximation of impulsive recurrent process with semi-Markov switching. Theory Probab. Math. Stat. **80**, 85–92 (2009). MR2777903

[8] Kosenkova, T.I.: Weak convergence of the series scheme of Markov chains to the solution of Lévy driven SDE. Theory Stoch. Process. **18(34)**(1), 86–100 (2012). MR3124765

[9] Skorokhod, A.V.: Studies in the Theory of Random Processes. Addison-Wesley, Reading (1965). MR0185620

[10] Strook, D.: Markov Processes from K. Itô's Perspective. Princeton University Press (2003)