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$\mathbb{L}^p (p \geq 2)$-solutions of generalized BSDEs with jumps and monotone generator in a general filtration

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Abstract In this paper, we study multidimensional generalized BSDEs that have a monotone generator in a general filtration supporting a Brownian motion and an independent Poisson random measure. First, we prove the existence and uniqueness of $\mathbb{L}^p(p \geq 2)$-solutions in the case of a fixed terminal time under suitable p-integrability conditions on the data. Then, we extend these results to the case of a random terminal time. Furthermore, we provide a comparison result in dimension 1.

Keywords Generalized backward stochastic differential equations (GBSDEs) with jumps, $\mathbb{L}^p$ solution, monotone generator, comparison theorem
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1 Introduction

This paper is concerned with the study of multidimensional generalized backward stochastic differential equations (GBSDEs) with jumps in a general filtration. For convenience of the discussion, let us first make precise the notion of such equations, which is adopted from [9].

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Let $T > 0$ be a fixed time horizon and consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathbb{P})$ carrying a standard d-dimensional Brownian motion W and an independent compensated Poisson random measure $\widehat{\pi}$. The filtration $(\mathcal{F}_t)_{0\le t\le T}$ is assumed to be complete and right continuous. Assume that we are given an $\mathbb{R}^k$-valued $\mathcal{F}_T$-measurable random variable ξ, a random function $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathcal{L}^2_\lambda \longrightarrow \mathbb{R}^k$ (see Section 2 for the definition of $\mathcal{L}^2_\lambda$) such that $f(\cdot, y, z, v)$ is $(\mathcal{F}_t)$-progressively measurable for each (y, z, v), and an $(\mathcal{F}_t)_{t\ge 0}$-progressively measurable càdlàg finite-variation process $(R_t)_{t\in[0,T]}$ such that $R_0 = 0$. Roughly speaking, solving a GBSDE with jumps in a general filtration with terminal time T associated with terminal condition ξ and generator $f + dR$ amounts to finding the usual triple (Y_t, Z_t, V_t) (with Y adapted and Z and V predictable) and a càdlàg martingale $M = (M_t)_{t\in[0,T]}$ that is orthogonal to W and $\widehat{\pi}$ (see Lemma 1) such that the following equation is satisfied $\mathbb{P}$-a.s.:

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds + \int_t^T dR_s - \int_t^T Z_s dW_s \\ - \int_t^T \int_U V_s(e)\widehat{\pi}(de, ds) - \int_t^T dM_s, \quad t \in [0, T]. \end{aligned} \tag{1}$$

This equation is usually denoted by GBSDE(ξ, $f + dR$). Note that the reason behind adding the martingale M to the definition of GBSDE (1) is the fact that we do not assume that the underlying filtration is generated by W and $\widehat{\pi}$, and in such cases, the martingale representation property may fail.

Nonlinear BSDEs with jumps (i.e., the underlying filtration is generated by a Brownian motion and an independent Poisson random measure) were first introduced by Tang and Li [19]. They proved the existence and uniqueness of the solution under a Lipschitz continuity condition on the generator w.r.t. the variables. Since then, a lot of papers (see, e.g., [13, 1, 17, 20, 16, 12, 9], and the references therein) and books (see, e.g., [18] and [5]) studied BSDEs with jumps due to the connections of this subject with mathematical finance (see, e.g., [5]) (e.g., if the Brownian motion stands for the noise from a financial market, then the Poisson random measure can be interpreted as the randomness of the insurance claims), stochastic control (see, e.g., [10]), and partial differential equations (see, e.g., [1]), and so on. Since the work of Tang and Li [19], the attempts of generalization of their results have been made in several different directions. First of all, many papers aimed at relaxing the Lipschitz condition on the generator w.r.t. y. For example, Pardoux [13] considered a monotonicity condition on the generator w.r.t. y and a linear growth condition on y. Some efforts were devoted to weaken the square integrability on the coefficients, for example, $\mathbb{E}[|\xi|^2 + \int_0^T |f(t, 0, 0, 0)|^2 dt] < +\infty$. Yao [20] gave the existence and uniqueness results for $\mathbb{L}^p$-solutions ($p > 1$) for BSDEs with jumps for a monotonic generator (not the same monotonicity condition considered in our paper) and $\mathbb{L}^p$ coefficients. Later, Li and Wei [12] analyzed fully coupled BSDEs with jumps and showed the existence and uniqueness of $\mathbb{L}^p$-solutions ($p \ge 2$) for such equations for a monotone generator and p-integrable data. Further, other settings of BSDEs with jumps have been introduced. Pardoux [13] studied a class of BSDEs with jumps called generalized BSDEs with jumps that involves an integral w.r.t. an increasing continuous process. The author shows the uniqueness and existence of the GBSDE with generator monotone in y and square-integrable data.

Recently, Kruse and Popier [9] considered another direction of generalization concerning the underlying filtration, which is no longer assumed to be generated by W and $\widehat{\pi}$. In fact, they studied multidimensional BSDEs in a general filtration of type (1) with $R \equiv 0$. The authors established the existence and uniqueness of $\mathbb{L}^p$-solutions ($p > 1$) under a monotonicity assumption on the generator f w.r.t. y and under the condition that the data ξ and $f(t, 0, 0, 0)$ are in $\mathbb{L}^p$, $p > 1$, that is,

$$\mathbb{E}\left[|\xi|^p + \int_0^T \left|f(t, 0, 0, 0)\right|^p dt \right] < +\infty. \tag{2}$$

Moreover, they also considered the case of a random terminal time that is not necessarily bounded.

In our paper, we first investigate the existence and uniqueness of $\mathbb{L}^p(p \geq 2)$-solutions (see Definition 1) for GBSDEs (1) in a deterministic time horizon. We suppose that f is monotonic w.r.t. y (this condition is essential in the study of BSDEs with random terminal time), Lipschitz continuous w.r.t. to z and v, and satisfies a very general growth condition w.r.t. y considered earlier in the Brownian setting in [4] and recently in the case of jumps in [9]:

$$\forall r > 0, \qquad \sup_{|y| \leq r} \left|f(\cdot, y, 0, 0) - f(\cdot, 0, 0, 0)\right| \in \mathbb{L}^1\big(\Omega \times [0, T]\big). \tag{3}$$

This condition seems to be the best possible growth condition on f w.r.t. y and is widely used in the theory of partial differential equations (see [2] and the references therein).

Concerning the data, we assume that a p-integrability condition is satisfied (see assumption $(H1)$). Moreover, under an additional assumption on the jump component of f (see $(H6')$ in Section 4), we provide a comparison principle in dimension one (see the counterexample in [1]). Then, we extend the results obtained in the case of deterministic time horizon to the case of a random terminal time that is not necessarily bounded.

Let us highlight the main contribution of the paper compared to the existing literature. On the one hand, our results extend the work of Kruse and Popier [9] to the case of generalized BSDEs. Furthermore, we strengthen their results even in the case $R = 0$ since our p-integrability condition on $f(t, 0, 0, 0)$ (see assumption $(H1)$) is weaker than the $\mathbb{L}^p$-integrability (2) assumed in their paper. It should be mentioned that, due to the p-integrability assumed on $f(t, 0, 0, 0)$ and also to the process dR, some difficulties arise. Indeed, as in [4] and [9], to study the $\mathbb{L}^p$-solutions, a result on the existence and uniqueness in the classical $\mathbb{L}^2$ case (see Theorem 1) is first needed. To obtain such a result, the main trick is to truncate the coefficients with suitable truncation functions in order to have a bounded solution Y, which is a key tool in the proof in the $\mathbb{L}^2$ case (see [14, Prop. 2.4 and Thm. 2.2], Proposition 2.2 in [3] used in [4, Thm. 4.2] and [9, Lemma 4 and Thm. 1]). The approach followed in the papers mentioned to get this important estimate fails in our context. This is the reason why we give nonstandard estimates in Lemma 2, which allow us to overcome this problem.

On the other hand, we generalize the work of Pardoux [13] to the situation of a general filtration. Moreover, even in the case of a Wiener–Poisson filtration (filtration generated by W and π), compared to [13], we weaken the growth condition on f w.r.t. y stated in assumption (3), instead of the linear growth condition on the variable y, and derive the existence and uniqueness of $\mathbb{L}^p$-solutions of our GBSDE, whereas only the classical $\mathbb{L}^2$-solutions were studied in [13]. Note also that, in our case, GBSDE involves an integral w.r.t. a finite-variation càdlàg process unlike [13], where an integral w.r.t. a continuous increasing process is considered instead.

Our main motivation for writing this paper is because it is a first step in the study of our future work on the existence and uniqueness of $\mathbb{L}^p$-solutions for reflected GBSDEs. Note that since we solve that problem using a penalization method, the comparison principle obtained here is primordial. Finally, to the best of our knowledge, there is no such result in the literature.

The rest of the paper is organized as follows: in the following section, we give the mathematical setting of this paper and some basic identities. In Section 3, we study the existence and uniqueness of $\mathbb{L}^p$-solutions on a fixed time interval, which is done in three parts. First, we study the classical case of $\mathbb{L}^2$-solutions. The proof method follows the arguments and techniques (convolution, weak convergence, truncation technique) given in [8, 13, 14, 4, 9] with obvious modifications. Then, in the remaining parts, we extend the result to $\mathbb{L}^p$-solutions for any $p \geq 2$, using the right a priori estimate, the $\mathbb{L}^2$ case result, and a truncation technique. In Section 4, a comparison principle for GBSDEs with jumps in dimension 1 is provided. Finally, Section 5 is devoted to the case of a random terminal time.

2 Preliminaries

Throughout this paper, $T > 0$ is a fixed time horizon, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ is a filtered probability space. The filtration $(\mathcal{F}_t, 0 \leq t \leq T)$ is assumed to be complete and right continuous. We suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ supports a d-dimensional Wiener process $(W_t, 0 \leq t \leq T)$ and a random Poisson measure π on $\mathbb{R}^+ \times U$, where $U := \mathbb{R}^n \backslash \{0\}$ is equipped with its Borel field $\mathcal{U}$, with the compensator $\nu(dt, de) = dt\lambda(de)$ such that $\{\widehat{\pi}([0, t] \times A) = (\pi - \nu)([0, t] \times A)\}_{t\leq T}$ is a martingale for all $A \in \mathcal{U}$ satisfying $\lambda(A) < +\infty$. Here, λ is assumed to be a σ-finite Lévy measure on $(U, \mathcal{U})$ such that

$$\int_U \left(1 \wedge |e|^2\right)\lambda(de) < \infty.$$

Let $\mathcal{P}$ denote the σ-algebra of predictable sets on $\Omega \times [0, T]$, and let us introduce the following notation:

- $\mathcal{S}$ is the set of all adapted càdlàg processes.
- $G_{loc}(\pi)$ is the set of $\mathcal{P} \times \mathcal{U}$-measurable functions V on $\Omega \times [0, T] \times U$ such that, for any $t \geq 0$,

$$\int_0^t \int_U \left(\left|V_s(e)\right|^2 \wedge \left|V_s(e)\right|\right)\lambda(de)(ds) < +\infty \quad \text{a.s.}$$

- $\mathcal{H}$ (resp. $\mathcal{H}(0,T)$) is the set of all predictable processes on $\mathbb{R}^+$ (resp. on $[0,T]$). $\mathbb{L}^2_{loc}(W)$ is the subspace of $\mathcal{H}$ such that, for any $t \geq 0$,

$$\int_0^t |Z_s|^2 ds < +\infty \quad \text{a.s.}$$

- $\mathbb{M}_{loc}$ is the set of càdlàg local martingales orthogonal to W and $\widehat{\pi}$. If $M \in \mathbb{M}_{loc}$, then

$$\left[M, W^i\right]_t = 0,\ 1 \leq i \leq d, \quad \text{and} \quad \left[M, \widehat{\pi}(A,\cdot)\right]_t = 0, \tag{4}$$

  for all $A \in \mathcal{U}$. In other words, $\mathbb{E}(\Delta M * \pi \mid \mathcal{P} \otimes \mathcal{U}) = 0$, where the product $*$ denotes the integral process (see II.1.5 in [7]).

- $\mathbb{M}$ is the subspace of $\mathbb{M}_{loc}$ of martingales.

- $\mathcal{V}$ is the set of all càdlàg progressively measurable processes R of finite variation such that $R_0 = 0$.

For a given process $R \in \mathcal{V}$, we denote by $|R|_t$ the variation of R on $[0,t]$ and by dR the random measure generated by its trajectories. By $\mathcal{T}$ we denote the set of all stopping times with values in $[0,T]$ and by $\mathcal{T}_t$ the set of all stopping times with values in $[t,T]$. We say that a sequence $(\tau_k)_{k\in\mathbb{N}} \subset T$ is stationary if $\mathbb{P}(\liminf_{k\to+\infty}\{\tau_k = T\}) = 1$. For $X \in \mathcal{S}$, we set $X_{t-} = \lim_{s\nearrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$ with the convention that $X_{0-} = 0$.

Now, since we are dealing with a general filtration, we recall Lemma III.4.24 in [7], which gives the representation property of a local martingale in our context.

Lemma 1. *Every local martingale has a decomposition*

$$\int_0^\cdot Z_s dW_s + \int_0^\cdot \int_{\mathcal{U}} V_s(e)\widehat{\pi}(de,ds) + M,$$

where $M \in \mathbb{M}_{loc}$, $Z \in \mathbb{L}^2_{loc}(W)$, *and* $V \in G_{loc}(\pi)$.

The Euclidean norm of a vector $y \in \mathbb{R}^k$ will be defined by $|y| = \sqrt{\sum_{i=1}^k |y_i|^2}$, and for any $k \times d$ matrix z, we define $|z| = \sqrt{\text{Trace}(zz^t)}$, where z^t stands for the transpose of z. The quadratic variation of a martingale $M \in \mathbb{R}^k$ is defined by $[M]_t = \sum_{i=1}^k [M^i]_t$. By $[M]^c$ we denote the continuous part of the quadratic variation $[M]$. Let us introduce the following spaces of processes for any real constant $p \geq 2$:

- $\mathbb{L}^p$ is the space of $\mathbb{R}^k$-valued random variables ξ such that

$$\|\xi\|_{\mathbb{L}^p} := E\left[|\xi|^p\right]^{1/p} < +\infty.$$

- $\mathcal{S}^p$ is the space of $\mathbb{R}^k$-valued, $\mathcal{F}_t$-adapted, and càdlàg processes $(Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathcal{S}^p} := E\Big[\sup_{0\leq t\leq T} |Y_t|^p\Big]^{1/p} < +\infty.$$

- $\mathbb{M}^p$ is the set of all $\mathbb{R}^k$-valued martingales $M \in \mathbb{M}$ such that $\mathbb{E}([M]_T)^{\frac{p}{2}} < +\infty$.
- $\mathcal{M}^p$ is the set of $\mathbb{R}^{k\times d}$-valued and $\mathcal{F}$-progressively measurable processes $(Z_t)_{0\le t\le T}$ such that

$$\|Z\|_{\mathcal{M}^p} := E\left[\left(\int_0^T |Z_s|^2 ds\right)^{\frac{p}{2}}\right]^{1/p} < +\infty.$$

- $\mathcal{L}^p$ is the set of $\mathcal{P}\otimes\mathcal{U}$-measurable mappings $V : \Omega\times[0,T]\times U \to \mathbb{R}^k$ such that

$$\left\|V(e)\right\|_{\mathcal{L}^p} := E\left[\left(\int_0^T \int_U \left|V_s(e)\right|^2 \lambda(de)ds\right)^{\frac{p}{2}}\right]^{1/p} < +\infty.$$

- $\mathcal{L}^p_\lambda$ is the set of measurable functions $\phi : U \to \mathbb{R}^k$ such that

$$\left\|\phi(e)\right\|_{\mathcal{L}^p_\lambda} := \left(\int_U \left|\phi(e)\right|^p \lambda(de)\right)^{1/p} < +\infty.$$

- Ξ^p is the space $\mathcal{S}^p\times\mathcal{M}^p\times\mathcal{L}^p\times\mathbb{M}^p$.
- $\mathcal{V}^p$ is the set of all processes $R\in\mathcal{V}$ such that $\|R\|_{\mathcal{V}^p} := \mathbb{E}(|R|_T^p)^{1/p} < \infty$, where $|R|_T$ denotes the total variation of R on $[0,T]$.

In what follows, let ξ be an $\mathbb{R}^k$-valued and $\mathcal{F}_T$-measurable random variable, and let R be a process in $\mathcal{V}$. Finally, let us consider a random function $f : [0,T]\times\Omega\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathcal{L}^2_\lambda \longrightarrow \mathbb{R}^k$ measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^k)\otimes\mathcal{B}(\mathbb{R}^{k\times d})\otimes\mathcal{B}(\mathcal{L}^2_\lambda)$. In the paper, we consider the following hypotheses:

$(H1)$ $\mathbb{E}[|\xi|^p + (\int_0^T |f(t,0,0,0)|dt)^p + |R|_T^p] < +\infty$.

$(H2)$ For every $(t,z,v)\in[0,T]\times\mathbb{R}^{k\times d}\times\mathcal{L}^2_\lambda$, the mapping $y\in\mathbb{R}^k \to f(t,y,z,v)$ is continuous.

$(H3)$ There exists $\mu\in\mathbb{R}$ such that

$$\big(f(t,y,z,v) - f\big(t,y',z,v\big)\big)\big(y-y'\big) \le \mu\big(y-y'\big)^2,$$

for all $t\in[0,T]$, $y,y'\in\mathbb{R}^k$, $z\in\mathbb{R}^{k\times d}$, $v\in\mathcal{L}^2_\lambda$.

$(H4)$ For every $r>0$, the mapping $t\in[0,T]\to\sup_{|y|\le r}|f(t,y,0,0)-f(t,0,0,0)|$ belongs to $\mathbb{L}^1(\Omega\times[0,T])$.

$(H5)$ f is Lipschitz continuous w.r.t. z, that is, there exists a constant $L>0$ such that

$$\left|f(t,y,z,v) - f\big(t,y,z',v\big)\right| \le L\left|z-z'\right|,$$

for all $t\in[0,T]$, $y\in\mathbb{R}^k$, $z,z'\in\mathbb{R}^{k\times d}$, $v\in\mathcal{L}^2_\lambda$.

$(H6)$ f is Lipschitz continuous w.r.t. v, that is, there exists a constant $L > 0$ such that

$$\left|f(t, y, z, v) - f\big(t, y, z, v'\big)\right| \leq L\left\|v - v'\right\|_{\mathcal{L}^2_\lambda},$$

for all $t \in [0, T]$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$, $v, v' \in \mathcal{L}^2_\lambda$.

To begin with, let us make precise the notion of $\mathbb{L}^p$-solutions of the GBSDE (1), which we consider throughout this paper.

Definition 1. We say that $(Y, Z, V, M) := (Y_t, Z_t, V_t, M_t)_{0\leq t\leq T}$ is an $\mathbb{L}^p$-solution of the GBSDE (1) if $(Y, Z, V, M) \in \Xi^p$ and (1) is satisfied $\mathbb{P}$-a.s.

3 Generalized BSDEs with constant terminal time

3.1 $\mathbb{L}^2$-solutions

In this subsection, we study the classical case of $\mathbb{L}^2$-solutions of GBSDE (1). The results given here generalize those of [13] and [9]. Note that the integrability condition $(H1)_{p=2}$ made on $f(\cdot, 0, 0, 0)$ is weaker than the assumption $E\int_0^T |f(t, 0, 0, 0)|^2 dt < +\infty$, $t \in [0, T]$, made in those papers, which means that our assumption is weaker than that of [9] even in the case $R \equiv 0$.

Let us begin by giving nonstandard a priori estimates on the solution, which will play a primordial role in the proof of Theorem 1. Let us first make the following assumption:

(A) There exist $L \geq 0$, $\mu \in \mathbb{R}$, and a nonnegative progressively measurable process $\{f_t\}_{t\in[0,T]}$ satisfying $\mathbb{E}(\int_0^T f_s ds)^2 < +\infty$ such that
$\forall (t, y, z, v) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathcal{L}^2_\lambda$,

$$\widehat{\operatorname{sgn}}(y) f(t, y, z, v) \leq f_t + \mu|y| + L|z| + L\|v\|_{\mathcal{L}^2_\lambda}, \quad dt \otimes d\mathbb{P}\text{-a.s.}$$

Remark 1. Note that (A) is not a new assumption, but a direct consequence of assumptions $(H3)$, $(H5)$, and $(H6)$ with $f_t = |f(t, 0, 0, 0)|$. In fact, three assumptions $(H3)$, $(H5)$, and $(H6)$ are reduced to a single one (assumption (A)) for simplicity.

Lemma 2. *Let assumption (A) hold, and let (Y, Z, V, M) be a solution of GBSDE (1). If $Y \in \mathcal{S}^2$ and*

$$\mathbb{E}|\xi|^2 + \mathbb{E}\left(\int_0^T f_s ds\right)^2 + \mathbb{E}|R|_T^2 < +\infty, \tag{5}$$

then, (Z, V, M) belongs to $\mathcal{M}^2 \times \mathcal{L}^2 \times \mathbb{M}^2$, and for some $a \geq \mu + 2L^2$, there is a constant $C > 0$ such that, for all $0 \leq q \leq t \leq T$,

$$\begin{aligned}\mathbb{E}\Bigg[\sup_{s\in[t,T]} e^{2as}|Y_s|^2 &+ \int_t^T e^{2as}|Z_s|^2 ds + \int_t^T \int_U e^{2as}\left|V_s(e)\right|^2 \lambda(de)ds \\ &+ e^{2aT}[M]_T - e^{2at}[M]_t \;\Big|\; \mathcal{F}_q\Bigg] \\ &\leq C\mathbb{E}\left[e^{2aT}|\xi|^2 + \left(\int_t^T e^{as} f_s ds\right)^2 + \left(\int_t^T e^{as} d|R|_s\right)^2 \;\Big|\; \mathcal{F}_q\right]. \end{aligned}\tag{6}$$

Proof. The proof is performed in two steps. For simplicity, we can assume w.l.o.g. that $a = 0$. Indeed, let us fix $a \geq \mu + 2L^2$ and define $\widetilde{Y}_t = e^{at} Y_t$, $\widetilde{Z} = e^{at} Z_t$, $\widetilde{V}_t = e^{at} V_t$, $d\widetilde{M}_t = e^{at} dM_t$. Observe that $(\widetilde{Y}, \widetilde{Z}, \widetilde{V}, \widetilde{M})$ solves the following GBSDE:

$$\begin{aligned}\widetilde{Y}_t = \widetilde{\xi} + \int_t^T \widetilde{f}(s, \widetilde{Y}_s, \widetilde{Z}_s, \widetilde{V}_s)ds + \int_t^T d\widetilde{R}_s - \int_t^T \widetilde{Z}_s dW_s \\ - \int_t^T \int_U \widetilde{V}_s(e)\widehat{\pi}(de, ds) - \int_t^T d\widetilde{M}_s, \quad t \in [0, T],\end{aligned}$$

where $\widetilde{\xi} = e^{aT}\xi$, $\widetilde{f}(t, y, z, v) = e^{at} f(t, e^{-at}y, e^{-at}z, e^{-at}v) - ay$, and $d\widetilde{R}_t = e^{at} dR_t$. Notice that $\widetilde{f}$ satisfies assumption (A) with $\widetilde{f}_t = e^{at} f_t$, $\widetilde{\mu} = \mu - a$, $\widetilde{L} = L$. Since we are working on a compact time interval, the integrability conditions are equivalent with or without the superscript $\sim$. Thus, with this change of variable, we reduce to the case $a = 0$ and $\mu + 2L^2 \leq 0$. We omit the superscript $\sim$ for notational convenience.

Step 1. First, we show that there exists a constant $C > 0$ such that, for all $0 \leq q \leq t \leq T$,

$$\begin{aligned}&\mathbb{E}\left(\int_t^T |Z_s|^2 ds + \int_t^T \int_U \left|V_s(e)\right|^2 \lambda(de)ds + \int_t^T d[M]_s \,\Big|\, \mathcal{F}_q\right) \\ &\quad \leq C\mathbb{E}\left(\sup_{u\in[t,T]} |Y_u|^2 + \left(\int_t^T f_s ds\right)^2 + \left(\int_t^T d|R|_s\right)^2 \,\Big|\, \mathcal{F}_q\right). \qquad (7)\end{aligned}$$

Since there is a lack of integrability of the processes (Z, V, M), we are proceeding by localization. For $n \in \mathbb{N}$, we set

$$\tau_n = \inf\left\{t > 0; \int_0^t |Z_s|^2 ds + \int_0^t \int_U \left|V_s(e)\right|^2 \lambda(de)ds + [M]_t > n\right\} \wedge T.$$

By Itô's formula (see [15, Thm. II.32]),

$$\begin{aligned}&|Y_{t\wedge\tau_n}|^2 + \int_{t\wedge\tau_n}^{\tau_n} |Z_s|^2 ds + \int_{t\wedge\tau_n}^{\tau_n} \int_U \left|V_s(e)\right|^2 \pi(de, ds) + \int_{t\wedge\tau_n}^{\tau_n} d[M]_s \\ &\quad = |Y_{\tau_n}|^2 + 2\int_{t\wedge\tau_n}^{\tau_n} Y_s f(s, Y_s, Z_s, V_s)ds + 2\int_{t\wedge\tau_n}^{\tau_n} Y_{s-} dR_s \\ &\qquad - 2\int_{t\wedge\tau_n}^{\tau_n} Y_s Z_s dW_s - 2\int_{t\wedge\tau_n}^{\tau_n} \int_U Y_{s-} V_s(e)\widehat{\pi}(de, ds) - 2\int_{t\wedge\tau_n}^{\tau_n} Y_{s-} dM_s. \qquad (8)\end{aligned}$$

But from (A), the basic inequality $2ab \leq 2a^2 + \frac{b^2}{2}$, and the fact that $\mu + 2L^2 \leq 0$ we have that

$$\begin{aligned}2Y_s f(s, Y_s, Z_s, V_s) &\leq 2L|Y_s||Z_s| + 2L|Y_s|\left\|V_s(e)\right\|_{\mathcal{L}^2_\lambda} + 2\mu|Y_s|^2 + 2|Y_s| f_s \\ &\leq 2\left(\mu + 2L^2\right)|Y_s|^2 + 2|Y_s| f_s + \frac{1}{2}|Z_s|^2 + \frac{1}{2}\int_U \left|V_s(e)\right|^2 \lambda(de) \\ &\leq 2|Y_s| f_s + \frac{1}{2}|Z_s|^2 + \frac{1}{2}\int_U \left|V_s(e)\right|^2 \lambda(de). \qquad (9)\end{aligned}$$

Then, plugging the last inequality into (8), we deduce

$$\begin{aligned}
&\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}|Z_s|^2ds+\int_{t\wedge\tau_n}^{\tau_n}\int_U\left|V_s(e)\right|^2\pi(de,ds)\\
&\quad-\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}\int_U\left|V_s(e)\right|^2\lambda(de)ds+\int_{t\wedge\tau_n}^{\tau_n}d[M]_s\\
&\leq \sup_{s\in[t\wedge\tau_n,T]}|Y_s|^2+2\sup_{s\in[t\wedge\tau_n,T]}|Y_s|\left(\int_{t\wedge\tau_n}^{T}f_sds\right)\\
&\quad+2\sup_{s\in[t\wedge\tau_n,T]}|Y_s|\left(\int_{t\wedge\tau_n}^{T}d|R|_s\right)-2\int_{t\wedge\tau_n}^{\tau_n}Y_sZ_sdW_s\\
&\quad-2\int_{t\wedge\tau_n}^{\tau_n}\int_U Y_{s-}V_s(e)\widehat{\pi}(de,ds)-2\int_{t\wedge\tau_n}^{\tau_n}Y_{s-}dM_s.
\end{aligned}$$

Hence, using the inequality $2ab \leq a^2+b^2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}|Z_s|^2ds+\int_{t\wedge\tau_n}^{\tau_n}\int_U\left|V_s(e)\right|^2\pi(de,ds)\\
&\quad-\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}\int_U\left|V_s(e)\right|^2\lambda(de)ds+\int_{t\wedge\tau_n}^{\tau_n}d[M]_s\\
&\leq 3\sup_{u\in[t\wedge\tau_n,T]}|Y_u|^2+\left(\int_{t\wedge\tau_n}^{T}f_sds\right)^2+\left(\int_{t\wedge\tau_n}^{T}d|R|_s\right)^2\\
&\quad-2\int_{t\wedge\tau_n}^{\tau_n}Y_sZ_sdW_s-2\int_{t\wedge\tau_n}^{\tau_n}\int_U Y_{s-}V_s(e)\widehat{\pi}(de,ds)-2\int_{t\wedge\tau_n}^{\tau_n}Y_{s-}dM_s. \quad (10)
\end{aligned}$$

Note that since $Y \in \mathcal{S}^2$, by the definition of the stopping time τ_n it follows by the BDG inequality that $\int_0^{t\wedge\tau_n}Y_sZ_sdW_s$, $\int_0^{t\wedge\tau_n}\int_U Y_{s-}V_s(e)\widehat{\pi}(de,ds)$ and $\int_0^{t\wedge\tau_n}Y_{s-}dM_s$ are uniformly integrable martingales. Consequently, taking the conditional expectation w.r.t. $\mathcal{F}_q$, $0\leq q\leq t\leq T$, in both sides of (10) yields

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}|Z_s|^2ds+\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}\int_U\left|V_s(e)\right|^2\lambda(de)ds+\int_{t\wedge\tau_n}^{\tau_n}d[M]_s\,\Big|\,\mathcal{F}_q\right)\\
&\quad\leq \mathbb{E}\left(3\sup_{u\in[t\wedge\tau_n,T]}|Y_u|^2+\left(\int_{t\wedge\tau_n}^{T}f_sds\right)^2+\left(\int_{t\wedge\tau_n}^{T}d|R|_s\right)^2\,\Big|\,\mathcal{F}_q\right).
\end{aligned}$$

Therefore, letting n to infinity and using Fatou's lemma, we obtain (7).

Step 2. In this step, we will estimate $\mathbb{E}(\sup_{u\in[t,T]}|Y_u|^2|\mathcal{F}_q)$, $0\leq q\leq t\leq T$. Applying Itô's formula to $|Y_t|^2$ for each $t\in[0,T]$, we get

$$\begin{aligned}
&|Y_t|^2+\int_t^T|Z_s|^2ds+\int_t^T\int_U\left|V_s(e)\right|^2\pi(de,ds)+\int_t^T d[M]_s\\
&\quad=|\xi|^2+2\int_t^T Y_sf(s,Y_s,Z_s,V_s)ds+2\int_t^T Y_{s-}dR_s\\
&\qquad-2\int_t^T Y_sZ_sdW_s-2\int_t^T\int_U Y_{s-}V_s(e)\widehat{\pi}(de,ds)-2\int_t^T Y_{s-}dM_s, \quad (11)
\end{aligned}$$

but in view of (9), we deduce

$$\begin{aligned}|Y_t|^2 &+ \frac{1}{2}\int_t^T |Z_s|^2 ds + \int_t^T \int_U \left|V_s(e)\right|^2 \pi(de,ds) + \int_t^T d[M]_s \\ &\le |\xi|^2 + 2\int_t^T |Y_s| f_s ds + 2\int_t^T |Y_{s-}| d|R|_s + \frac{1}{2}\int_t^T \int_U \left|V_s(e)\right|^2 \lambda(de) ds \\ &\quad - 2\int_t^T Y_s Z_s dW_s - 2\int_t^T \int_U Y_{s-} V_s(e) \widehat{\pi}(de,ds) - 2\int_t^T Y_{s-} dM_s. \qquad (12)\end{aligned}$$

Recalling that $Y \in \mathcal{S}^2$, thanks to the first step (estimate (7)), it follows that $Z \in \mathcal{M}^2$, $V \in \mathcal{L}^2$, and $M \in \mathbb{M}^2$. Therefore, by the BDG inequality we deduce that $\int_0^t Y_s Z_s dW_s$, $\int_0^t \int_U Y_{s-} V_s(e) \widehat{\pi}(de,ds)$, and $\int_0^t Y_{s-} dM_s$ are uniformly integrable martingales. Therefore, taking the conditional expectation in (12) w.r.t. $\mathcal{F}_q$, it follows that, for all $0 \le q \le t \le T$,

$$\mathbb{E}\left(\frac{1}{2}\int_t^T |Z_s|^2\, ds + \frac{1}{2}\int_t^T \int_U \left|V_s(e)\right|^2 \lambda(de) ds + \int_t^T d[M]_s \,\Big|\, \mathcal{F}_q\right) \le \mathbb{E}(\varsigma \mid \mathcal{F}_q), \qquad (13)$$

where $\varsigma = |\xi|^2 + 2\int_t^T |Y_s| f_s ds + 2\int_t^T |Y_{s-}| d|R|_s$.

Next, we deduce from (12) that

$$\begin{aligned}\sup_{u\in[t,T]} |Y_u|^2 &\le \varsigma + \frac{1}{2}\int_t^T \int_U \left|V_s(e)\right|^2 \lambda(de) ds + 2\sup_{u\in[t,T]}\left|\int_u^T Y_s Z_s dW_s\right| \\ &\quad + 2\sup_{u\in[t,T]}\left|\int_u^T \int_U Y_{s-} V_s(e) \widehat{\pi}(de,ds)\right| + 2\sup_{u\in[t,T]}\left|\int_u^T Y_{s^-} dM_s\right|. \qquad (14)\end{aligned}$$

Consequently, taking the conditional expectation w.r.t. $\mathcal{F}_q$, $0 \le q \le t \le T$, we obtain from (13) that

$$\begin{aligned}&\mathbb{E}\big(\sup_{u\in[t,T]} |Y_u|^2 \mid \mathcal{F}_q\big) \\ &\le \mathbb{E}\left(2\varsigma + 2\sup_{u\in[t,T]}\left|\int_u^T Y_s Z_s dW_s\right| + 2\sup_{u\in[t,T]}\left|\int_u^T \int_U Y_{s-} V_s(e) \widehat{\pi}(de,ds)\right|\right. \\ &\quad \left. + 2\sup_{u\in[t,T]}\left|\int_u^T Y_{s^-} dM_s\right| \,\Big|\, \mathcal{F}_q\right). \qquad (15)\end{aligned}$$

Next, applying the BDG inequality to the martingale terms implies that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}&2\mathbb{E}\left(\sup_{u\in[t,T]}\left|\int_u^T Y_s Z_s dW_s\right| \,\Big|\, \mathcal{F}_q\right) \\ &\le 2C_1 \mathbb{E}\left(\sup_{u\in[t,T]} |Y_u| \left(\int_t^T |Z_s|^2 ds\right)^{1/2} \,\Big|\, \mathcal{F}_q\right)\end{aligned}$$

$$\leq \frac{1}{4}\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|^2 \;\Big|\; \mathcal{F}_q\Big) + 2C_1^2\mathbb{E}\Big(\int_t^T |Z_s|^2 ds \;\Big|\; \mathcal{F}_q\Big), \tag{16}$$

$$\begin{aligned}
&2\mathbb{E}\Big(\sup_{u\in[t,T]}\Big|\int_u^T Y_{s-}V_s(e)\widehat{\pi}(de,ds)\Big| \;\Big|\; \mathcal{F}_q\Big)\\
&\quad\leq 2C_1\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|\Big(\int_t^T \big|V_s(e)\big|^2\pi(de,ds)\Big)^{1/2} \;\Big|\; \mathcal{F}_q\Big)\\
&\quad\leq \frac{1}{4}\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|^2 \;\Big|\; \mathcal{F}_q\Big) + 2C_1^2\mathbb{E}\Big(\int_t^T \big|V_s(e)\big|^2\lambda(de)ds \;\Big|\; \mathcal{F}_q\Big),
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&2\mathbb{E}\Big(\sup_{u\in[t,T]}\Big|\int_u^T Y_{s-}dM_s\Big| \;\Big|\; \mathcal{F}_q\Big)\\
&\quad\leq 2C_1\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|\Big(\int_t^T d[M]_s\Big)^{1/2} \;\Big|\; \mathcal{F}_q\Big)\\
&\quad\leq \frac{1}{4}\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|^2 \;\Big|\; \mathcal{F}_q\Big) + 2C_1^2\mathbb{E}\Big(\int_t^T d[M]_s \;\Big|\; \mathcal{F}_q\Big).
\end{aligned} \tag{18}$$

Hence, plugging estimates (16)–(18) into (15) implies in view of (13) that there exits a constant $C_2 > 0$ such that

$$\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|^2 \;\Big|\; \mathcal{F}_q\Big) \leq C_2\mathbb{E}(\varsigma \mid \mathcal{F}_q).$$

Applying Young's inequality yields

$$\begin{aligned}
&C_2\mathbb{E}\Big(\int_t^T |Y_s|f_s ds + \int_t^T |Y_{s-}|d|R|_s \;\Big|\; \mathcal{F}_q\Big)\\
&\quad\leq \frac{1}{4}\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|^2 \;\Big|\; \mathcal{F}_q\Big) + C_3\mathbb{E}\Big[\Big(\int_t^T f_s ds\Big)^2 + \Big(\int_t^T d|R|_s\Big)^2 \;\Big|\; \mathcal{F}_q\Big],
\end{aligned} \tag{19}$$

from which we deduce, coming back to the definition of ς, that there exists $C_4 > 0$ such that

$$\mathbb{E}\Big(\sup_{u\in[t,T]}|Y_u|^2 \;\Big|\; \mathcal{F}_q\Big) \leq C_4\mathbb{E}\Big(|\xi|^2 + \Big(\int_t^T f_s ds\Big)^2 + \Big(\int_t^T d|R|_s\Big)^2 \;\Big|\; \mathcal{F}_q\Big).$$

Finally, combining this with (7), the desired result follows, which ends the proof. □

Now, we give the main result of this subsection.

Theorem 1. (L^2-*solutions*) *Assume that* $(H1)_{p=2}$–$(H6)$ *are in force. Then, there exists a unique* $\mathbb{L}^2$-*solution* (Y, Z, V, M) *for the GBSDE* (1).

Proof. *Uniqueness*. Let (Y, Z, V, M) and (Y', Z', V', M') denote respectively two $\mathbb{L}^2$-solutions of GRBSDE (1). Define $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M}) = (Y - Y', Z - Z', V - V', M - M')$. Then, $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M})$ solves the following GBSDE in Ξ^2:

$$\bar{Y}_t = \int_t^T \big(f(s, Y_s, Z_s, V_s) - f\big(s, Y'_s, Z'_s, V'_s\big)\big)ds - \int_t^T \bar{Z}_s dW_s \qquad (20)$$
$$- \int_t^T \int_U \bar{V}_s(e)\widehat{\pi}(de, ds) - \int_t^T d\bar{M}_s, \quad t \in [0, T].$$

It follows from $(H3)$, $(H5)$, and $(H6)$ that

$$\begin{aligned}
&\widehat{\text{sgn}}(\bar{y})\big(f(t, y, z, v) - f\big(t, y', z', v'\big)\big) \\
&\quad = \widehat{\text{sgn}}(\bar{y})\big[f(t, y, z, v) - f\big(t, y', z, v\big) + f\big(t, y', z, v\big) - f\big(t, y', z', v\big) \\
&\qquad + f\big(t, y', z', v\big) - f\big(t, y', z', v'\big)\big] \\
&\quad \le \mu|y| + L|z| + L\|v\|_{\mathcal{L}^2_\lambda},
\end{aligned}$$

which means that assumption (A) is satisfied for the generator of GBSDE (20) with $f_t \equiv 0$. By Lemma 2 with $q = t = 0$ we obtain immediately that $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M}) = (0, 0, 0, 0)$. The proof of the uniqueness is then complete.

Existence. Before giving the proof of the existence part, we will talk a little bit about it, but let us first give the following assumption, the so-called general growth condition, which will be needed later:

(H_{gg}) For every $(t, y) \in [0, T] \times \mathbb{R}$, $\quad \big|f(t, y, 0, 0)\big| \le \big|f(t, 0, 0, 0)\big| + \gamma\big(|y|\big),$ (21)

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a deterministic continuous increasing function.

The proof method of Theorem 1 is enlightened by [8, 13, 14, 4, 9], but of course with some obvious changes. More precisely, the first step uses arguments given in [8, 13, 9], whereas the techniques used in the second step, the convolution and weak convergence, are borrowed from [13]. The truncation techniques applied in the third and fourth steps are taken partly from [4, 9, 8]. However, it should be mentioned that since we have changed, compared to [9], the $\mathbb{L}^2$-integrability condition of $f(t, 0, 0, 0)$ from $\mathbb{E}\int_0^T |f(t, 0, 0, 0)|^2 dt < +\infty$ to the one given in $(H1)_{p=2}$ and due also to the finite-variation part dR, some new troubles come up, especially, when we want to prove an analogous result of [9, Lemma 4], which says that whenever the data is bounded, so is the solution of the GBSDE. Their approach fails in our context. This is the reason why we give nonstandard estimates in Lemma 2, which allows us to overcome this problem (see, e.g., estimates (26) and (28)). Additionally, in order to prove the existence part of Theorem 1, we first need an existence result under the assumptions of this theorem but with $(H4)$ replaced with (H_{gg}), which extends results given in [13] and [9].

The proof is divided into five steps as follows. Note that we will frequently apply Lemma 2. For simplicity, we will assume w.l.o.g. that $a = 0$, which means that, in this case, $\mu < 0$ (since $\mu + 2L^2 \le a = 0$, i.e., $\mu \le -2L^2 < 0$). In the rest of the proof, we will assume that $\mu < 0$.

Step 1. We first assume additionally that there exists a constant $l > 0$ such that

$$\big|f(t, y, z, v) - f\big(t, y', z, v\big)\big| \le l\big|y - y'\big|, \qquad (22)$$

for $t \in [0, T]$, $y, y' \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$, $v \in \mathcal{L}^2_\lambda$. Moreover, we assume also that there exists a constant $\epsilon > 0$ such that

$$|\xi| + \sup_{t\in[0,T]} \left|f(t, 0, 0, 0)\right| + |R|_T \leq \epsilon. \tag{23}$$

For $(\Gamma, \Upsilon, \Psi, N) \in \Xi^2$, in view of the assumptions made on ξ, f, and R, define the processes (Y, Z, V, M) as follows:

$$\begin{aligned} Y_t = \mathbb{E}\left[\xi + \int_0^T f(s, \Gamma_s, \Upsilon_s, \Psi_s)ds + \int_0^T dR_s \,\middle|\, \mathcal{F}_t\right] \\ - \int_0^t f(s, \Gamma_s, \Upsilon_s, \Psi_s)ds - \int_0^t dR_s, \end{aligned}$$

and the local martingale

$$\mathbb{E}\left[\xi + \int_0^T f(s, \Gamma_s, \Upsilon_s, \Psi_s)ds + \int_0^T dR_s \,\middle|\, \mathcal{F}_t\right] - Y_0,$$

which thanks to the martingale representation theorem (see Lemma 1), can be decomposed as follows:

$$\int_0^t Z_s dW_s + \int_0^t \int_U V_s(e)\widehat{\pi}(de, ds) + M_t,$$

where Z, V, and M belong respectively to $\mathbb{L}^2_{loc}(W)$, $G_{loc}(\pi)$, and $\mathbb{M}_{loc}$. Therefore, (Y, Z, V, M) is the unique solution of the GBSDE

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, \Gamma_s, \Upsilon_s, \Psi_s)ds + \int_t^T dR_s - \int_t^T Z_s dW_s \\ - \int_t^T \int_U V_s(e)\widehat{\pi}(de, ds) - \int_t^T dM_s, \quad t \in [0, T]. \end{aligned} \tag{24}$$

Moreover, from the conditions on ξ, f and R it is easy to prove that $(Y, Z, V, M) \in \Xi^2$.

As a by-product, we may define the mapping $\Phi : \Xi^2 \to \Xi^2$ that associates $(\Gamma, \Upsilon, \Psi, N)$ with $\Phi((\Gamma, \Upsilon, \Psi, N)) = (Y, Z, V, M)$. By standard arguments (see, e.g., the proof of [13, Thm. 55.1]) it can be shown that Φ is contractive on the Banach space Ξ^2 endowed with the norm

$$\begin{aligned} \left\|(Y, Z, V, M)\right\|_\beta = \mathbb{E}\Bigg\{\sup_{t\in[0,T]} e^{\beta t}|Y_t|^2 + \int_0^T e^{\beta t}|Z_t|^2 dt \\ + \int_0^T \int_U e^{\beta t}\left|V_t(e)\right|^2 \lambda(de)dt + \left[\int_0^\cdot e^{\beta t} dM_t\right]_T\Bigg\}^{\frac{1}{2}}, \end{aligned}$$

for a suitably chosen constant $\beta > 0$. Consequently, Φ has a fixed point $(Y, Z, V, M) \in \Xi^2$. Therefore, clearly, (Y, Z, V, M) is the unique solution of GBSDE (1) under the assumptions made so far.

Step 2. In this step, we will show how to dispense with assumption (22). We state and prove the following lemma.

Lemma 3. *Assume that* $(H1)_{p=2}$, $(H2)$, $(H3)$, (H_{gg}), $(H5)$, $(H6)$, *and* (23) *hold. For given* $(\Upsilon, \Psi) \in \mathcal{M}^2 \times \mathcal{L}^2$, *there exists a unique quadruple of processes* $(Y, Z, V, M) \in \Xi^2$ *such that*

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, Y_s, \Upsilon_s, \Psi_s)ds + \int_t^T dR_s - \int_t^T Z_s dW_s \\ - \int_t^T \int_U V_s(e)\widehat{\pi}(de, ds) - \int_t^T dM_s. \end{aligned} \tag{25}$$

For notational convenience, we set $f(t, y) = f(t, y, \Upsilon_t, \Psi_t)$ for each $y \in \mathbb{R}^k$.

Proof. Uniqueness is proved by arguing as for uniqueness in Theorem 1. The result follows immediately. For the existence part, we follow the line of the proof of [14, Prop. 2.4]. Now, let us assume that (23) holds and define $f_n(t, y) = (\rho_n * f(t, \cdot))(y)$, where $\rho_n : \mathbb{R} \to \mathbb{R}^+$ is a sequence of smooth functions with compact support that approximate the Dirac measure at 0 and satisfy $\int \rho_n(z)dz = 1$. Moreover, they are defined such that ϖ satisfying $\varpi(r) = \sup_n \sup_{|y|\le r} \int_{\mathbb{R}} \gamma(|y|)\rho_n(y - z)dz$ is finite for all $r \in \mathbb{R}^+$. Note that f satisfies the following assumptions:

(*i*) $|f(t, y)| \le |f(t, 0, 0, 0| + L(|\Upsilon_s| + \|\Psi_s\|_{\mathcal{L}^2_\lambda}) + \gamma(|y|)$,

(*ii*) $\mathbb{E}\int_0^T |f(t, 0)|^2 dt < +\infty$,

(*iii*) $(y - y')(f(t, y) - f(t, y')) \le 0$,

(*iv*) $y \to f(t, y)$ is continuous for all t a.s.

Thus, it is elementary to check that f_n satisfies (i)–(iv) with the same constant L and ϖ instead of γ. However, we cannot apply step 1 of the proof since f_n is not necessarily globally Lipschitz continuous in y but only locally Lipschitz. Hence, to overcome this problem, we add a truncation function T_p in f_n. Indeed, define, for each $p \in \mathbb{N}$,

$$f_{n,p}(t, y) = f_n\big(t, T_p(y)\big) \quad \text{such that } T_p(y) = \frac{py}{|y| \vee p}.$$

Notice that, for all $n, p \in \mathbb{N}$, $y \to f_{n,p}(t, y)$ is globally Lipschitz and satisfies the conditions of Step 1. Therefore, for all $n, p \in \mathbb{N}$, according to what has already been proved in Step 1, there exists a unique solution $(Y^{n,p}, Z^{n,p}, V^{n,p}, M^{n,p})$ to GBSDE (25) associated with $(\xi, f_{n,p} + dR)$. Furthermore, it follows from (23) and Lemma 2 with $a = 0$ and $q = t$ that there exists a universal constant $C_1 > 0$ such that, for all $n, p \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} &\big|Y_t^{n,p}\big|^2 + \mathbb{E}\bigg(\int_t^T \big|Z_s^{n,p}\big|^2 ds + \int_t^T \int_U \big|V_s^{n,p}(e)\big|^2 \lambda(de)ds + \int_t^T d\big[M^{n,p}\big]_s \,\bigg|\, \mathcal{F}_t\bigg) \\ &\le C_1 \mathbb{E}\bigg[|\xi|^2 + \bigg(\int_t^T \big|f(s, 0, 0, 0)\big|ds\bigg)^2 + \bigg(\int_t^T d|R|_s\bigg)^2 \,\bigg|\, \mathcal{F}_t\bigg] \\ &\le C_1 \epsilon^2\big(2 + T^2\big) := r^2. \end{aligned} \tag{26}$$

Hence, for any $p > r$, the sequence $(Y^{n,p}, Z^{n,p}, V^{n,p}, M^{n,p})$ does not depend on p. Then we denote it by (Y^n, Z^n, V^n, M^n), and it is a solution to GBSDE (25) associated with $(\xi, f_n + dR)$. Moreover, now f_n satisfies the conditions of Lemma 2 with a constant independent of n, and thus the sequence $(Y^n, U^n, Z^n, V^n, M^n)$ is uniformly bounded, that is,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\bigg[\int_0^T |Y_t^n|^2 dt + \Big(\int_0^T f_n\big(t, Y_t^n\big)dt\Big)^2 + \int_0^T |Z_t^n|^2 dt \\ + \int_0^T \int_U \big|V_t^n(e)\big|^2 \lambda(de)dt + \big[M^n\big]_T\bigg] \leq C. \tag{27}$$

Let us set $U_t^n = f_n(t, Y_t^n)$ for $t \in [0, T]$. Using the previous uniform estimate of the sequence $\{(Y^n, Z^n, V^n, U^n,)\}_n$ and the Hilbert structure of $\mathbb{L}^2(\Omega \times [0, T]) \times \mathcal{M}^2 \times \mathcal{L}^2 \times \mathfrak{H}^2$, we deduce that we can extract subsequences, still denoted $\{n\}$, that weakly converge to some process (Y, Z, V, U) in $\mathbb{L}^2(\Omega \times [0, T]) \times \mathcal{M}^2 \times \mathcal{L}^2 \times \mathfrak{H}^2$, where $\mathfrak{H}^2$ denotes the set of $\mathcal{F}_t$-progressively measurable $\mathbb{R}^k$-valued processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{\mathfrak{H}^2} := \bigg\{\mathbb{E}\bigg[\Big(\int_0^T |U_t|dt\Big)^2\bigg]\bigg\}^{\frac{1}{2}} < +\infty.$$

Now we deal with the convergence of the martingale M^n. By estimate (27) it follows that $\sup_{n\geq 1} \mathbb{E}|M_T^n|^2 < \infty$. Thus, there exists a subsequence, still denoted $\{n\}$, such that M_T^n converges weakly to some random variable M_T in $\mathbb{L}^2(\Omega)$. Let M_t denote the martingale with terminal value M_T.

Next, following [14, Prop. 2.4], we deduce, by using the martingale representation theorem (see Lemma 1) and orthogonality that the following weak convergence hold for the martingales in $\mathbb{L}^2(\Omega)$: for each $t \in [0, T]$,

$$\int_t^T Z_s^n dW_s \to \int_t^T Z_s dW_s, \int_t^T \int_U V_s^n(e)\hat{\pi}(de, ds) \to \int_t^T \int_U V_s(e)\hat{\pi}(de, ds), \\ \text{and} \quad M_t^n \to M_t.$$

Therefore, taking weak limits in the approximating equation, we get that (Y, Z, V, M) satisfies the GBSDE

$$Y_t = \xi + \int_t^T U_s ds + \int_t^T dR_s - \int_t^T Z_s dW_s - \int_t^T \int_U V_s(e)\widehat{\pi}(de, ds) - \int_t^T dM_s.$$

Finally, as in [14, Prop. 2.4], we can show that $U_t = f(t, Y_t)$. This implies that (Y, Z, V, M) solves GBSDE (25) under the assumptions made so far in this step, which completes the proof of Lemma 3.

Step 3. In this step, we will show that assumption (H_{gg}) assumed so far can be weakened to $(H4)$. In fact, by the mean of truncation technique we will show that, for given $(\Upsilon, \Psi) \in \mathcal{M}^2 \times \mathcal{L}^2$ and provided that $(H1)_{p=2}$–$(H6)$ and (23) hold, GBSDE (25) has a solution in Ξ^2. The idea behind the proof is to approximate f by a sequence of functions f_n satisfying assumption (H_{gg}). Indeed, let θ_r be a smooth function such that $0 \le \theta_r \le 1$ and satisfies for r large enough:

$$\theta_r(y) = \begin{cases} 1 & \text{for} \quad |y| \le r, \\ 0 & \text{for} \quad |y| \ge r+1. \end{cases}$$

Let $\varphi_r(t) = \sup_{|y|\le r} |f(t, y, 0, 0) - f(t, 0, 0, 0)| \in \mathbb{L}^1([0, T])$ and, for $n \in \mathbb{N}^*$, denote $T_n(x) = \frac{xn}{|x|\vee n}$. The approximation sequence f_n is defined by

$$f_n(t, y, \Upsilon, \Psi) = \big(f\big(t, y, T_n(\Upsilon_t), T_n(\Psi_t)\big) - f(t, 0, \Upsilon_t, \Psi_t)\big)\frac{n}{\varphi_{r+1}(t) \vee n} + f(t, 0, \Upsilon_t, \Psi_t).$$

We also define a sequence h_n that truncates f_n for $|y| \ge r+1$:

$$h_n(t, y, \Upsilon_t, \Psi_t) = \theta_r(y)\big(f\big(t, y, T_n(\Upsilon_t), T_n(\Psi_t)\big) - f(t, 0, \Upsilon_t, \Psi_t)\big)\frac{n}{\varphi_{r+1}(t) \vee n} + f(t, 0, \Upsilon_t, \Psi_t).$$

Following the same reasoning as in the proof of [4, Thm. 4.2], it can be shown that h_n still satisfies the monotonicity condition $(H3)$ but with a positive constant $C(r, k, n)$ depending on r, k, and n. Then, the conditions of Lemma 3 of the previous step are fulfilled by the data $(\xi, h_n + dR)$. Consequently, for each $n \in \mathbb{N}^*$, the GBSDE (25) associated with $(\xi, h_n + dR)$, admits a unique solution $(Y^n, Z^n, V^n, M^n) \in \Xi^2$. Moreover, since $yh_n(t, y, \Upsilon, \Psi) \le |y|\|f(t, 0, 0, 0)\|_\infty + k|y|(|\Upsilon| + \|\Psi\|_{\mathcal{L}^2_\lambda})$, h_n satisfies the condition of Lemma 2. Consequently, applying Lemma 2 with $a = 0$ and $q = t = 0$, in view of the boundedness assumption (23), we get similarly as in (26) that, for each $n \in \mathbb{N}$, the following estimates hold $d\mathbb{P} \times dt$-a.e.:

$$\left|Y_t^n\right| \le r \quad \text{and} \quad \mathbb{E}\left(\int_0^T \left|Z_s^n\right|^2 ds + \int_0^T \int_U \left|V_s^n(e)\right|^2 \lambda(de)ds + \left[M^n\right]_T\right) \le r^2. \quad (28)$$

As a by-product, (Y^n, Z^n, V^n, M^n) is a solution to the GBSDE (25) associated with the data $(\xi, f_n + dR)$. Next, we show as in [9, Thm. 1] (see also [4, Thm. 4.2]) by using similar arguments that (Y^n, Z^n, V^n, M^n) is a Cauchy sequence in Ξ^2, and its limit is $(Y, Z, V, M) \in \Xi^2$.

Step 4. We now treat the general case. We want to get rid of the boundedness condition (23) used so far. To this end, a truncation procedure. Indeed, under assumptions $(H1)_{p=2}$–$(H6)$, we first set, for each $n \in \mathbb{N}^*$,

$$\xi_n = T_n(\xi), \qquad f_n(t, y) = f_n(t, y, \Upsilon_t, \Psi_t) = f(t, y) - f(t, 0) + T_n\big(f(t, 0)\big),$$
$$R_t^n = \int_0^t \mathbb{1}_{\{|R|_s \le n\}} dR_s.$$

Then, according to the previous step, for each $n \in \mathbb{N}^*$, GBSDE (25) associated with $(\xi_n, f_n + dR^n)$ has a unique solution $(Y^n, Z^n, V^n, M^n) \in \Xi^2$. Our goal now is to show that (Y^n, Z^n, V^n, M^n) is a Cauchy sequence in Ξ^2. Set $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M}) =$

$(Y^m - Y^n, Z^m - Z^n, V^m - V^n, M^m - M^n)$. Then $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M})$ is solution to the GBSDE

$$\bar{Y}_t = \xi_m - \xi_n + \int_t^T d\big(R_s^m - R_s^n\big) + \int_t^T \big(f_m\big(s, Y_s^m\big) - f_n\big(s, Y_s^n\big)\big)ds - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_U \bar{V}_s(e)\widehat{\pi}(de, ds) - \int_t^T d\bar{M}_s, \quad t \in [0, T]. \tag{29}$$

Thanks to $(H3)$, $(H5)$, and $(H6)$, the generator of GBSDE (29) satisfies assumption (A) with $f_t \equiv T_m(f(t,0)) - T_n(f(t,0))$ and $L = 0$. Therefore, applying Lemma 2 with $a = 0$ and $q = t = 0$ yields, for all $n, m \in \mathbb{N}$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\bar{Y}_t|^2 + \int_0^T |\bar{Z}_t|^2 dt + \int_0^T \int_U \big|\bar{V}_t(e)\big|^2 \lambda(de)dt + [\bar{M}]_T\Big] \leq C\mathbb{E}\Big[|\xi_m - \xi_n|^2 + \Big(\int_0^T \big|T_m\big(f(t,0)\big) - T_n\big(f(t,0)\big)\big|dt\Big)^2 + \Big(\int_0^T d\big|R^m - R^n\big|_s\Big)^2\Big]. \tag{30}$$

Obviously, the right-hand side of (30) tends to 0 as $n, m \to \infty$. Therefore, (Y^n, Z^n, V^n, M^n) is a Cauchy sequence in Ξ^2, and its limit $(Y, Z, V, M) \in \Xi^2$ is an L^2-solution of GBSDE (25).

Step 5. In this step, we will finally complete the proof of the existence part of Theorem 1. To this end, we consider a Picard's iteration procedure. Set $(Y^0, Z^0, V^0, M^0) = (0, 0, 0, 0)$ and define $\{(Y_t^n, Z_t^n, V_t^n, M_t^n)_{t\in[0,T]}\}_{n\geq 1}$ recursively in view of Step 4, for all $n \geq 0$ and $t \in [0, T]$,

$$Y_t^{n+1} = \xi + \int_t^T f\big(s, Y_s^{n+1}, Z_s^n, V_s^n\big)ds + \int_t^T dR_s - \int_t^T Z_s^{n+1} dW_s - \int_t^T \int_U V_s^{n+1}(e)\widehat{\pi}(de, ds) - \int_t^T dM_s^{n+1}. \tag{31}$$

From the previous step it follows that, under assumptions $(H1)_{p=2}$–$(H6)$, for each $n \geq 0$, there exists a solution of GBSDE (31). Let us set $\delta Y^n := Y^{n+1} - Y^n$, $\delta Z^n := Z^{n+1} - Z^n$, $\delta V^n := V^{n+1} - V^n$, and $\delta M^n := M^{n+1} - M^n$. $(\delta Y^n, \delta Z^n, \delta V^n, \delta M^n)$ solves the GBSDE

$$\delta Y_t^n = \int_t^T \big[f\big(s, Y_s^{n+1}, Z_s^n, V_s^n\big) - f\big(s, Y_s^n, Z_s^{n-1}, V_s^{n-1}\big)\big]ds - \int_t^T \delta Z_s^n dW_s - \int_t^T \int_U \delta V_s^n(e)\widehat{\pi}(de, ds) - \int_t^T d\delta M_s^n, \quad t \in [0, T].$$

By assumptions $(H3)$, $(H5)$, and $(H6)$ we have that

$$\begin{aligned} &\widehat{\text{sgn}}\big(\delta y^n\big)\big(f\big(t, y^{n+1}, z^n, v^n\big) - f\big(t, y^n, z^{n-1}, v^{n-1}\big)\big) \\ &\quad = \widehat{\text{sgn}}\big(\delta y^n\big)\big[\big(f\big(t, y^{n+1}, z^n, v^n\big) - f\big(t, y^n, z^n, v^n\big)\big) \end{aligned}$$

$$+\big(f\big(t,y^n,z^n,v^n\big)-f\big(t,y^n,z^{n-1},v^{n-1}\big)\big)\big]$$
$$\leq \mu\big|\delta y^n\big|+L\big|\delta z^{n-1}\big|+L\big\|\delta v^{n-1}\big\|_{\mathcal{L}^2_\lambda},$$

which is assumption (A) with $f_t = L|\delta z^{n-1}| + L\|\delta v^{n-1}\|_{\mathcal{L}^2_\lambda}$ and $L \equiv 0$. Thus, it follows from Lemma 2 with $a = q = t = 0$ and Hölder's inequality that there exists a constant $C > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\delta Y_t^n\big|^2+\int_0^T\big|\delta Z_s^n\big|^2ds+\int_0^T\int_U\big|\delta V_s^n(e)\big|^2\lambda(de)ds+\big[\delta M^n\big]_T\Big]$$
$$\leq C\mathbb{E}\Big[\int_0^T L\big(\big|\delta Z_t^{n-1}\big|+\big\|\delta V_t^{n-1}(e)\big\|_{\mathcal{L}^2_\lambda}\big)dt\Big]^2$$
$$\leq CL^2T\mathbb{E}\Big[\int_0^T\big(\big|\delta Z_t^{n-1}\big|^2+\big\|\delta V_t^{n-1}(e)\big\|^2_{\mathcal{L}^2_\lambda}\big)dt\Big].$$

Consequently, by induction we deduce that, for $n \geq 2$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\delta Y_t^n\big|^2+\int_0^T\big|\delta Z_t^n\big|^2dt+\int_0^T\int_U\big|\delta V_t^n(e)\big|^2\lambda(de)dt+\big[\delta M^n\big]_T\Big]$$
$$\leq c^{n-1}\mathbb{E}\Big[\int_0^T\big|\delta Z_t^1\big|^2dt+\int_0^T\int_U\big|\delta V_t^1(e)\big|^2\lambda(de)dt\Big],$$

where $c = CL^2T$. Let us first assume, for a sufficiently small T, that $c < 1$. Then, since the remaining term of the right-hand side of the last inequality is finite, we deduce that (Y^n, Z^n, V^n, M^n) is a Cauchy sequence in Ξ^2, and the limit process (Y, Z, V, M) is a solution to GBSDE (1) in Ξ^2.

For the general case, it suffices to subdivide the interval time $[0, T]$ into a finite number of small intervals, and using the standard arguments, we can prove the existence of a solution (Y, Z, V, M) of GBSDE (1) in Ξ^2 on the whole interval $[0, T]$. This completes the proof of this step and thus of the whole proof of Theorem 1. □

3.2 Case $p \geq 2$

In this subsection, we study the issue of existence and uniqueness of $\mathbb{L}^p$-solutions of GBSDE (1) in the case $p \geq 2$. Let us first give a priori estimates for the solution and their variations induced by a variation of the data.

Lemma 4. *Let assumption (A) hold, and let (Y, Z, V, M) be a solution of GBSDE (1). Let us assume moreover that*

$$\mathbb{E}\Big[|\xi|^p+\Big(\int_0^T f_t dt\Big)^p+|R|_T^p\Big]<+\infty. \tag{32}$$

If $Y \in S^p$, then there exists a constant $C_p > 0$, depending only on p and T, such that, for every $a \geq \mu + 2L^2$,

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} e^{apt}|Y_t|^p + \Big(\int_0^T e^{2at}|Z_t|^2 dt\Big)^{p/2} + \Big(\int_0^T \int_U e^{2at}\big|V_t(e)\big|^2 \lambda(de)dt\Big)^{p/2} + e^{apT}[M]_T^{p/2}\bigg] \leq C\mathbb{E}\bigg[e^{apT}|\xi|^p + \Big(\int_0^T e^{at} f_t dt\Big)^p + \Big(\int_0^T e^{at} d|R|_t\Big)^p\bigg]. \quad (33)$$

Proof. The proof is divided into two steps. By an already used argument (see Lemma (2)) we can assume w.l.o.g. that $\mu + 2L^2 \leq 0$ and take $a = 0$.

Step 1. First, we show that

$$\mathbb{E}\bigg[\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}} + \Big(\int_0^T \int_U \big|V_s(e)\big|^2 \lambda(de)ds\Big)^{\frac{p}{2}} + \Big(\int_0^T d[M]_s\Big)^{\frac{p}{2}}\bigg] \leq C(p,T)\bigg[\mathbb{E}\sup_{t\in[0,T]} |Y_t|^p + \Big(\int_0^T f_s ds\Big)^p + \Big(\int_0^T d|R|_s\Big)^p\bigg]. \quad (34)$$

Indeed, define the sequence of stopping times τ_n for $n \in \mathbb{N}$:

$$\tau_n = \inf\Big\{t > 0; \int_0^t |Z_s|^2 ds + \int_0^t \int_U \big|V_s(e)\big|^2 \lambda de(ds) + [M]_t > n\Big\} \wedge T.$$

By Itô's formula on $|Y_t|^2$,

$$|Y_0|^2 + \int_0^{\tau_n} |Z_s|^2 ds + \int_0^{\tau_n} \int_U \big|V_s(e)\big|^2 \pi(de,ds) + \int_0^{\tau_n} d[M]_s = |Y_{\tau_n}|^2 + 2\int_0^{\tau_n} Y_s f(s,Y_s,Z_s,V_s)ds + 2\int_0^{\tau_n} Y_{s-}dR_s - 2\int_0^{\tau_n} Y_s Z_s dW_s - 2\int_0^{\tau_n}\int_U Y_{s-}V_s(e)\widehat{\pi}(de,ds) - 2\int_0^{\tau_n} Y_{s-}dM_s.$$

But from assumption (A), combined with the inequality $2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ for $\varepsilon > 0$, since $\mu + 2L^2 < 0$, we have

$$\begin{aligned} 2Y_s f(s,Y_s,Z_s,V_s) &\leq 2\mu|Y_s|^2 + 2|Y_s||f_s| + 2L|Y_s||Z_s| + 2L|Y_s|\big\|V_s(e)\big\|_{\mathcal{L}^2_\lambda} \\ &\leq \Big(\frac{1}{\varepsilon} + 2\big(\mu + L^2\big)\Big)|Y_s| + 2|Y_s|f_s + \frac{1}{2}|Z_s|^2 + \varepsilon\big\|V_s(e)\big\|^2_{\mathcal{L}^2_\lambda} \\ &\leq \frac{1}{\varepsilon}|Y_s|^2 + 2|Y_s|f_s + \frac{1}{2}|Z_s|^2 + \varepsilon\big\|V_s(e)\big\|^2_{\mathcal{L}^2_\lambda}. \end{aligned} \quad (35)$$

Thus, since $\tau_n \leq T$, we deduce that

$$\frac{1}{2}\int_0^{\tau_n} |Z_s|^2 ds + \int_0^{\tau_n}\int_U \big|V_s(e)\big|^2 \pi(de,ds) + \int_0^{\tau_n} d[M]_s \leq \sup_{t\in[0,T]} |Y_t|^2 + \frac{1}{\varepsilon}\int_0^T |Y_s|^2 ds + 2\sup_{t\in[0,T]} |Y_t|\int_0^T f_s ds$$

$$
\begin{aligned}
&+2\sup_{t\in[0,T]}|Y_t|\int_0^T d|R|_s+\varepsilon\int_0^{\tau_n}\int_U \big|V_s(e)\big|^2\lambda(de)ds+2\bigg|\int_0^{\tau_n} Y_sZ_sdW_s\bigg|\\
&+2\bigg|\int_0^{\tau_n}\int_U Y_{s-}V_s(e)^2\widehat{\pi}(de,ds)\bigg|+2\bigg|\int_0^{\tau_n} Y_{s-}dM_s\bigg|\\
\le&\left(3+\frac{T}{\epsilon}\right)\sup_{t\in[0,T]}|Y_t|^2+\left(\int_0^T f_sds\right)^2+\left(\int_0^T d|R|_s\right)^2\\
&+\varepsilon\int_0^{\tau_n}\int_U \big|V_s(e)\big|^2\lambda(de)ds+\bigg|\int_0^{\tau_n} Y_sZ_sdW_s\bigg|\\
&+\bigg|\int_0^{\tau_n}\int_U Y_{s-}V_s(e)^2\widehat{\pi}(de,ds)\bigg|+\bigg|\int_0^{\tau_n} Y_{s-}dM_s\bigg|.
\end{aligned}
$$

It follows that there exists a constant $c_p > 0$, depending only on p, such that

$$
\begin{aligned}
&\left(\int_0^{\tau_n}|Z_s|^2ds\right)^{\frac{p}{2}}+\left(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\pi(de,ds)\right)^{\frac{p}{2}}+\left(\int_0^{\tau_n}d[M]_s\right)^{\frac{p}{2}}\\
&\quad\le c_p\Bigg[\left(3+\frac{T}{\epsilon}\right)^{\frac{p}{2}}\sup_{t\in[0,T]}|Y_t|^p+\left(\int_0^T f_sds\right)^p+\left(\int_0^{\tau_n}d|R|_s\right)^p\\
&\qquad+\epsilon^{\frac{p}{2}}\left(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\lambda(de)ds\right)^{\frac{p}{2}}+\bigg|\int_0^{\tau_n}Y_sZ_sdW_s\bigg|^{\frac{p}{2}}\\
&\qquad+\bigg|\int_0^{\tau_n}\int_U Y_{s-}V_s(e)^2\widehat{\pi}(de,ds)\bigg|^{\frac{p}{2}}+\bigg|\int_0^{\tau_n}Y_{s-}dM_s\bigg|^{\frac{p}{2}}\Bigg]. \qquad (36)
\end{aligned}
$$

Since $\frac{p}{2} \geq 1$, we can apply the BDG inequality to obtain

$$
\begin{aligned}
c_p\mathbb{E}\bigg|\int_0^{\tau_n}Y_sZ_sdW_s\bigg|^{\frac{p}{2}}&\le d_p\mathbb{E}\left[\left(\int_0^{\tau_n}|Y_s|^2|Z_s|^2ds\right)^{\frac{p}{4}}\right]\\
&\le\frac{d_p^2}{4}\mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^p\Big)+\frac{1}{2}\mathbb{E}\left(\int_0^{\tau_n}|Z_s|^2ds\right)^{\frac{p}{2}}, \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
c_p\mathbb{E}\bigg|\int_0^{\tau_n}Y_{s-}dM_s\bigg|^{\frac{p}{2}}&\le d_p\mathbb{E}\left[\left(\int_0^{\tau_n}|Y_{s-}|^2d[M]_s\right)^{\frac{p}{4}}\right]\\
&\le\frac{d_p^2}{4}\mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^p\Big)+\frac{1}{2}\mathbb{E}[M]_{\tau_n}^{\frac{p}{2}}, \qquad (38)
\end{aligned}
$$

and

$$
\begin{aligned}
&c_p\mathbb{E}\bigg|\int_0^T\int_U Y_{s-}V_s(e)\widehat{\pi}(de,ds)\bigg|^{\frac{p}{2}}\\
&\quad\le d_p\mathbb{E}\left[\left(\int_0^{\tau_n}|Y_s|^2\big|V_s(e)\big|^2\pi(de,ds)\right)^{\frac{p}{4}}\right]
\end{aligned}
$$

$$\leq \frac{d_p^2}{4}\mathbb{E}\Big(\sup_{t\in[0,T]}|Y_t|^p\Big) + \frac{1}{2}\mathbb{E}\Big(\int_0^{\tau_n}\big|V_s(e)\big|^2\pi(de,ds)\Big)^{\frac{p}{2}}. \tag{39}$$

Plugging estimates (37)–(39) into (36) and then taking the expectation, we get

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\Big(\int_0^{\tau_n}|Z_s|^2 ds\Big)^{\frac{p}{2}} + \Big(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\pi(de,ds)\Big)^{\frac{p}{2}} + \Big(\int_0^{\tau_n}d[M]_s\Big)^{\frac{p}{2}}\Big]\\
&\quad\leq C(p,T,\epsilon)\mathbb{E}\sup_{t\in[0,T]}|Y_t|^p + c_p\mathbb{E}\Big(\int_0^T f_s ds\Big)^p + c_p\mathbb{E}\Big(\int_0^{\tau_n}d|R|_s\Big)^p\\
&\qquad + c_p\epsilon^{\frac{p}{2}}\mathbb{E}\Big[\Big(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\lambda(de)ds\Big)^{\frac{p}{2}}\Big].
\end{aligned}$$

By [11, Section 4] or [6, Lemma 2.1] we have that, for some constant $\eta_p > 0$,

$$\mathbb{E}\Big(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\lambda(de)ds\Big)^{\frac{p}{2}} \leq \eta_p\mathbb{E}\Big(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\pi(de,ds)\Big)^{\frac{p}{2}}. \tag{40}$$

Thus, choosing ϵ small enough and depending only on p, we deduce that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^{\tau_n}|Z_s|^2 ds\Big)^{\frac{p}{2}} + \Big(\int_0^{\tau_n}\int_U\big|V_s(e)\big|^2\lambda(de)ds\Big)^{\frac{p}{2}} + \Big(\int_0^{\tau_n}d[M]_s\Big)^{\frac{p}{2}}\Big]\\
&\quad\leq \widetilde{C}(p,T)\mathbb{E}\sup_{t\in[0,T]}|Y_t|^p + \widetilde{C}_p\mathbb{E}\Big[\Big(\int_0^T f_s ds\Big)^p + \Big(\int_0^T d|R|_s\Big)^p\Big].
\end{aligned}$$

Finally, letting n to $+\infty$ and using Fatou's lemma, (34) follows.

Step 2. Since $p \geq 2$, we can apply Itô's formula with the $\mathcal{C}^2$ function $|y|^p$ to $|Y_t|^p$. Note that

$$\frac{\partial\theta}{\partial y_i}(y) = p y_i|y|^{p-2}, \qquad \frac{\partial^2\theta}{\partial y_i\partial y_j}(y) = p|y|^{p-2}\delta_{i,j} + p(p-2)y_i y_j|y|^{p-4},$$

where $\delta_{i,j}$ is the Kronecker delta. Thus, for every $t \in [0,T]$, we have

$$\begin{aligned}
|Y_t|^p &= |\xi|^p + p\int_t^T Y_{s-}|Y_{s-}|^{p-2}dR_s + p\int_t^T Y_s|Y_s|^{p-2}f(s,Y_s,Z_s,V_s)ds\\
&\quad - p\int_t^T Y_{s-}|Y_{s-}|^{p-2}dM_s - p\int_t^T Y_s|Y_s|^{p-2}Z_s dW_s\\
&\quad - p\int_t^T\int_U\big(Y_{s-}|Y_{s-}|^{p-2}V_s(e)\big)\widehat{\pi}(de,ds) - \frac{1}{2}\int_t^T \mathrm{Trace}\big(D^2\theta(Y_s)Z_sZ_s^t\big)ds\\
&\quad - \int_t^T\int_U\big(\big|Y_{s-}+V_s(e)\big|^p - |Y_{s-}|^p - pY_{s-}|Y_{s-}|^{p-2}V_s(e)\big)\pi(de,ds) - \aleph_t,
\end{aligned} \tag{41}$$

where

$$\aleph_t = \frac{1}{2}\int_t^T \sum_{1\le i,j\le d} \frac{\partial^2\theta}{\partial y_i \partial y_j}(Y_s)d\big[M^i, M^j\big]_s^c$$
$$+ \sum_{t<s\le T} \big(|Y_{s-} + \Delta M_s|^p - |Y_{s-}|^p - pY_{s-}|Y_{s-}|^{p-2}\Delta M_s\big).$$

Following arguments from [9, Prop. 2], we have that

$$\mathrm{Trace}\big(D^2\theta(y)zz^t\big) \ge p|y|^{p-2}|z|^2, \tag{42}$$

$$\aleph_t \ge \alpha_p \int_t^T |Y_{s-}|d[M]_s, \tag{43}$$

and

$$-\int_t^T\int_U \big(\big|Y_{s-} + V_s(e)\big|^p - |Y_{s-}|^p - pY_{s-}|Y_{s-}|^{p-2}V_s(e)\big)\pi(de, ds)$$
$$\le -p(p-1)3^{1-p}\int_t^T |Y_{s-}|^{p-2}\big|V_s(e)\big|^2\pi(de, ds), \tag{44}$$

where $\alpha_p = \min(\frac{p}{2}, p(p-1)3^{1-p})$.

Consequently, in view of estimates (42), (43), and (44), Eq. (41) becomes

$$|Y_t|^p + \alpha_p\int_t^T |Y_s|^{p-2}|Z|_s^2 ds + \alpha_p\int_t^T\int_U |Y_s|^{p-2}\big|V_s(e)\big|^2\pi(de, ds)$$
$$+ \alpha_p\int_t^T |Y_s|^{p-2}d[M]_s^c + \alpha_p\sum_{t<s\le T}|Y_s|^{p-2}|\Delta M_s|^2$$
$$\le |\xi|^p + p\int_t^T Y_{s-}|Y_{s-}|^{p-2}dR_s + p\int_t^T Y_s|Y_s|^{p-2}f(s, Y_s, Z_s, V_s)ds$$
$$- p\int_t^T Y_{s-}|Y_{s-}|^{p-2}dM_s - p\int_t^T Y_s|Y_s|^{p-2}Z_s dW_s$$
$$- p\int_t^T\int_U Y_{s-}|Y_{s-}|^{p-2}V_s(e)\,\widehat{\pi}(de, ds). \tag{45}$$

But from assumption (A) and the fact that $\mu \le -2L^2 \le 0$ (since $\mu + 2L^2 \le 0$) we deduce by using the inequality $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ that

$$Y_s f(s, Y_s, Z_s, V_s) \le \frac{L^2}{\epsilon}|Y_s|^2 + |Y_s|f_s + \frac{\epsilon}{2}|Z_s|^2 + \frac{\epsilon}{2}\int_U \big|V_s(e)\big|^2\lambda(de). \tag{46}$$

Choosing $\epsilon = \frac{\alpha_p}{p}$, we obtain in view of the last inequality that

$$|Y_t|^p + \frac{\alpha_p}{2}\int_t^T |Y_s|^{p-2}|Z|_s^2 ds + \alpha_p\int_t^T\int_U |Y_s|^{p-2}\big|V_s(e)\big|^2\pi(de, ds)$$
$$+ \alpha_p\int_t^T |Y_s|^{p-2}d[M]_s$$

$$\leq |\xi|^p + \frac{pL^2}{\alpha_p}\int_t^T |Y_s|^p ds + p\int_t^T |Y_s|^{p-1} f_s ds + p\int_t^T |Y_{s-}|^{p-1} dR_s$$
$$+ \frac{\alpha_p}{2}\int_t^T |Y_s|^{p-2}\|V_s(e)\|^2_{\mathcal{L}^2_\lambda} ds - p\int_t^T Y_s|Y_s|^{p-2} Z_s dW_s$$
$$- p\int_t^T\int_U Y_{s-}|Y_{s-}|^{p-2} V_s(e)\widehat{\pi}(de, ds) - p\int_t^T Y_{s-}|Y_{s-}|^{p-2} dM_s. \quad (47)$$

Let us set $X = |\xi|^p + \frac{pL^2}{\alpha_p}\int_t^T |Y_s|^p\, ds + p\int_t^T |Y_s|^{p-1}|f_s|ds + p\int_t^T |Y_{s-}|^{p-1} dR_s$, $\Upsilon_t = \int_0^t Y_s|Y_s|^{p-2} Z_s dW_s$, $\Theta_t = \int_0^t\int_U Y_{s-}|Y_{s-}|^{p-2} V_s(e)\widehat{\pi}(de, ds)$, and $\Gamma_t = \int_0^t Y_{s-}|Y_{s-}|^{p-2} dM_s$.

It follows from the BDG inequality that Υ_t, Θ_t, and Γ_t are uniformly integrable martingales. Indeed, by Young's inequality we have

$$\mathbb{E}\big([\Upsilon]_T^{\frac{1}{2}}\big) \leq \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^{p-1}\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{1}{2}}\Big]$$
$$\leq \frac{p-1}{p}\mathbb{E}\Big(\sup_{t\in[0,T]} |Y_t|^p\Big) + \frac{1}{p}\mathbb{E}\Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}}, \quad (48)$$

$$\mathbb{E}\big([\Theta]_T^{\frac{1}{2}}\big) \leq \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^{p-1}\Big(\int_0^T \big|V_s(e)\big|^2 \pi(de, ds)\Big)^{\frac{1}{2}}\Big]$$
$$\leq \frac{p-1}{p}\mathbb{E}\Big(\sup_{t\in[0,T]} |Y_t|^p\Big) + \frac{1}{p}\mathbb{E}\Big(\int_0^T \|V_s(e)\|^2_{\mathcal{L}^2_\lambda} ds\Big)^{\frac{p}{2}}, \quad (49)$$

and

$$\mathbb{E}\big([\Gamma]_T^{\frac{1}{2}}\big) \leq \mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t|^{p-1}[M]_T^{\frac{1}{2}}\Big] \leq \frac{p-1}{p}\mathbb{E}\Big(\sup_{t\in[0,T]} |Y_t|^p\Big) + \frac{1}{p}\mathbb{E}[M]_T^{\frac{p}{2}}. \quad (50)$$

The claim holds since the last terms of (48), (49), and (50) are finite. This is due to the fact that $Y \in \mathcal{S}^p$, which implies by the first step of the proof that $Z \in \mathcal{M}^p$, $V \in \mathcal{L}^p$, and $M \in \mathbb{M}^p$. Moreover, we have

$$\mathbb{E}\int_t^T\int_U |Y_s|^{p-2}\big|V_s(e)\big|^2 \pi(de, ds) = \mathbb{E}\int_t^T |Y_s|^{p-2}\|V_s(e)\|^2_{\mathcal{L}^2_\lambda} ds. \quad (51)$$

Hence, in view of (51), taking the expectation in (47) yields

$$\frac{\alpha_p}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}|Z|_s^2 ds + \frac{\alpha_p}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\|V_s(e)\|^2_{\mathcal{L}^2_\lambda} ds$$
$$+ \alpha_p\mathbb{E}\int_t^T |Y_s|^{p-2} d[M]_s$$
$$\leq \mathbb{E}[X]. \quad (52)$$

Furthermore, coming back to (47), we deduce in view of (52) that

$$\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p \le 2\mathbb{E}X + p\mathbb{E}\bigg(\sup_{s\in[t,T]}\bigg|\int_s^T\int_U Y_{u^-}|Y_{u^-}|^{p-2}V_u(e)\widehat{\pi}(de,du)\bigg|\bigg)$$
$$+ p\mathbb{E}\bigg(\sup_{s\in[t,T]}\bigg|\int_s^T Y_u|Y_u|^{p-2}Z_u dW_u\bigg|\bigg)$$
$$+ p\mathbb{E}\bigg(\sup_{s\in[t,T]}\bigg|\int_s^T Y_{u^-}|Y_{u^-}|^{p-2}dM_u\bigg|\bigg). \tag{53}$$

The BDG inequality implies that

$$p\mathbb{E}\bigg(\sup_{s\in[t,T]}\bigg|\int_s^T Y_u|Y_u|^{p-2}Z_u dW_u\bigg|\bigg)$$
$$\le d_p\mathbb{E}\bigg[\bigg(\int_t^T |Y_s|^{2p-2}|Z_s|^2 ds\bigg)^{\frac{1}{2}}\bigg]$$
$$\le \frac{1}{4}\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p + 4d_p^2\mathbb{E}\bigg(\int_t^T |Y_s|^{p-2}|Z_s|^2 du\bigg), \tag{54}$$

$$p\mathbb{E}\bigg(\sup_{s\in[t,T]}\bigg|\int_s^T\int_U Y_{u^-}|Y_{u^-}|^{p-2}V_u(e)\widehat{\pi}(de,du)\bigg|\bigg)$$
$$\le d_p\mathbb{E}\bigg(\int_t^T\int_U |Y_u|^{2p-2}\big|V_s(e)\big|^2\pi(de,ds)\bigg)^{\frac{1}{2}}$$
$$\le \frac{1}{4}\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p + 4d_p^2\mathbb{E}\bigg(\int_t^T |Y_u|^{p-2}\big\|V(e)_u\big\|^2_{\mathcal{L}^2_\lambda} du\bigg), \tag{55}$$

and

$$p\mathbb{E}\bigg(\sup_{s\in[t,T]}\bigg|\int_s^T Y_{u^-}|Y_{u^-}|^{p-2}dM_u\bigg|\bigg)$$
$$\le d_p\mathbb{E}\big(|Y_s|^{2p-2}d[M]_s\big)^{\frac{1}{2}}$$
$$\le \frac{1}{4}\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p + 4d_p^2\mathbb{E}\bigg(\int_t^T |Y_s|^{p-2}d[M]_s\bigg). \tag{56}$$

Thus, combining estimates (54)–(56) with (52), we deduce that

$$\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p \le C_p\mathbb{E}[X]. \tag{57}$$

But, applying Young's inequality, we get

$$pC_p\mathbb{E}\int_t^T |Y_s|^{p-1}|f_s|ds \le pC_p\mathbb{E}\bigg(\sup_{s\in[t,T]}|Y_t|^{p-1}\int_t^T f_s ds\bigg)$$
$$\le \frac{1}{6}\mathbb{E}\sup_{s\in[t,T]}|Y_t|^p + d_p'\mathbb{E}\bigg(\int_t^T f_s ds\bigg)^p,$$

and

$$pC_p\mathbb{E}\int_t^T |Y_{s-}|^{p-1}dR_s \leq \frac{1}{6}E\sup_{s\in[t,T]}|Y_t|^p + d''_p\mathbb{E}\Big(\int_t^T d|R|_s\Big)^p.$$

Consequently, rearranging (57) in view of the two last estimates implies

$$\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p \leq C'_p\mathbb{E}\Big[|\xi|^p + \Big(\int_t^T f_s ds\Big)^p + \Big(\int_t^T d|R|_s\Big)^p\Big] + C''_p\int_t^T \mathbb{E}\sup_{u\in[s,T]}|Y_u|^p ds, \quad t\in[0,T].$$

Finally, using Gronwall's lemma, we deduce that

$$\mathbb{E}\sup_{t\in[0,T]}|Y_t|^p \leq C'_p e^{C''_p T}\mathbb{E}\Big[|\xi|^p + \Big(\int_0^T f_s ds\Big)^p + \Big(\int_0^T d|R|_s\Big)^p\Big]. \tag{58}$$

This, combined with (34), ends the proof. □

Lemma 5. *Let* (ξ, f, R) *and* (ξ', f', R') *be two sets of data, each satisfying assumptions* $(H1)$–$(H6)$. *Let* (Y, Z, V, M) *and* (Y', Z', V', M') *denote respectively an* $\mathbb{L}^p$*-solution of GRBSDE* (1) *with data* (ξ, f, R) *and* (ξ', f', R'). *Define*

$$(\bar{Y}, \bar{Z}, \bar{V}, \bar{M}, \bar{\xi}, \bar{f}, \bar{R}) = \big(Y - Y', Z - Z', V - V', M - M', \xi - \xi', f - f', R - R'\big).$$

Then there exists a constant $C > 0$, *depending on* p *and* T, *such that, for every* $a \geq \mu + 2L^2$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} e^{apt}|\bar{Y}_t|^p + \Big(\int_0^T e^{2at}|\bar{Z}_t|^2 dt\Big)^{\frac{p}{2}} + \Big(\int_0^T\int_U e^{2at}|\bar{V}_t|^2\lambda(de)dt\Big)^{\frac{p}{2}} + e^{apT}[\bar{M}]_T^{\frac{p}{2}}\Big]$$
$$\leq C\mathbb{E}\Big[e^{apT}|\bar{\xi}|^p + \Big(\int_0^T e^{at}\big|\bar{f}\big(s, Y'_t, Z'_t, V'_t\big)\big|dt\Big)^p + \Big(\int_0^T e^{at}d|\bar{R}|_t\Big)^p\Big]. \tag{59}$$

Proof. By an already used change-of-variable argument we may assume that $a = 0$. Obviously, $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M})$ solves the following GBSDE in Ξ^p:

$$\bar{Y}_t = \bar{\xi} + \int_t^T \big(f(s, Y_s, Z_s, V_s) - f'\big(s, Y'_s, Z'_s, V'_s\big)\big)ds + \int_t^T d\bar{R}_s - \int_t^T \bar{Z}_s dW_s - \int_t^T\int_U \bar{V}_s(e)\widehat{\pi}(de, ds) - \int_t^T d\bar{M}_s, \quad t\in[0,T]. \tag{60}$$

It follows from $(H3)$, $(H5)$, and $(H6)$ that

$$\begin{aligned}
&\widehat{\text{sgn}}(\bar{y})\big(f(t, y, z, v) - f'\big(t, y', z', v'\big)\big)\\
&= \widehat{\text{sgn}}(\bar{y})\big(f(t, y, z, v) - f\big(t, y', z', v'\big)\big) + \widehat{\text{sgn}}(\bar{y})\bar{f}\big(t, y', z', v'\big)\\
&= \widehat{\text{sgn}}(\bar{y})\big[f(t, y, z, v) - f\big(t, y', z, v\big) + f\big(t, y', z, v\big) - f\big(t, y', z', v\big)\\
&\quad + f\big(t, y', z', v\big) - f\big(t, y', z', v'\big)\big] + \widehat{\text{sgn}}(\bar{y})\bar{f}\big(t, y', z', v'\big)\\
&\leq \big|\bar{f}\big(t, y', z', v'\big)\big| + \mu|y| + L|z| + L\|v\|_{\mathcal{L}^2_\lambda},
\end{aligned} \tag{61}$$

which means that assumption (A) is satisfied for the generator of GBSDE (60), with $f_t \equiv |\bar{f}(t, y', z', v')|$. Thus, by Lemma 4 the desired estimate follows, which ends the proof of Lemma 5. □

Now we are able to give the main result of this subsection, the existence and uniqueness of an L^p-solution of GBSDE (1) in the case $p \geq 2$.

Theorem 2. *Let $p \geq 2$ and assume that $(H1)$–$(H6)$ hold. Then, there exists a unique $\mathbb{L}^p$-solution (Y, Z, V, M) for the GBSDE (1).*

Proof. Uniqueness follows immediately from Lemma 5. Now we deal with the existence. Set $T_n(x) = \frac{xn}{|x| \vee n}$ for $n \in \mathbb{N}^*$ and define ξ_n, f_n, and R^n as follows:

$$\xi_n = T_n(\xi), \qquad f_n(t, y, z, v) = f(t, y, z, v) - f(t, 0, 0, 0) + T_n\big(f(t, 0, 0, 0)\big),$$
$$R^n_t = \int_0^t \mathbb{1}_{\{|R|_s \leq n\}} dR_s.$$

Let (Y^n, Z^n, V^n, M^n) be a solution of GBSDE (1) associated with $(\xi_n, f_n + dR_n)$. Hence, by Theorem 1, for every $n \in \mathbb{N}$, there exists a unique solution to $(Y^n, Z^n, V^n, M^n) \in \varXi^2$ of GBSDE (1) associated with $(\xi^n, f_n + dR^n)$, but in fact also in $\varXi^p$, $p \geq 2$, according to Lemma 4. Our goal now is to show that (Y^n, Z^n, V^n, M^n) is a Cauchy sequence in $\varXi^p$. For $m \geq n$, applying Lemma 5 yields

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{t\in[0,T]} \big|Y^m_t - Y^n_t\big|^2 + \left(\int_0^T \big|Z^m_t - Z^n_t\big|^2 dt\right)^{\frac{p}{2}} \\
&\quad + \left(\int_0^T \int_U \big|V^m_t(e) - V^n_t(e)\big|^2 \lambda(de) dt\right)^{\frac{p}{2}} + \big[M^m - M^n\big]_T^{\frac{p}{2}}\Bigg] \\
&\leq C\mathbb{E}\Bigg[|\xi_m - \xi_n|^p + \left(\int_0^T \big|T_m\big(f(t, 0, 0, 0)\big) - T_n\big(f(t, 0, 0, 0)\big)\big| dt\right)^p \\
&\quad + \left(\int_0^T d\big|R^m - R^n\big|_s\right)^p\Bigg].
\end{aligned}$$

Therefore, letting n and m to infinity, we conclude that (Y^n, Z^n, V^n, M^n) is a Cauchy sequence in $\varXi^p$ and its limit $(Y, Z, V, M) \in \varXi^p$ is a solution of GBSDE (1) associated with $(\xi, f + dR)$, which ends the proof. □

4 Comparison theorem

In this section, we assume that $k = 1$ and aim at showing a comparison theorem for GBSDE. Our result, in particular, extends to the case of generalized BSDEs in a general filtration the comparison theorem given in [9, Prop. 4]. We follow the argument of [16]. In particular, we consider the Doléans–Dade exponential local martingale. Let α, β be predictable processes integrable w.r.t. dt and dW_t, respectively. Let γ be a predictable process defined on $[0, T] \times \Omega \times \mathbb{R}$ integrable w.r.t. $\widehat{\pi}(de, ds)$. For any $0 \leq t \leq s \leq T$, let E be the solution of

$$dE_{t,s} = E_{t,s-}\left[\beta_s dW_s + \int_U \gamma_s(e) \widehat{\pi}(de, ds)\right], \quad E_{t,t} = 1,$$

and let Γ be the solution of

$$d\Gamma_{t,s} = \Gamma_{t,s-}\Big[\alpha_s ds + \beta_s dW_s + \int_U \gamma_s(e)\widehat{\pi}(de,ds)\Big], \quad \Gamma_{t,t} = 1. \tag{62}$$

Of course, $\Gamma_{t,s} = \exp(\int_t^s \alpha_r dr)E_{t,s}$, and

$$E_{t,s} = \exp\bigg(\int_t^s \beta_r dW_r - \frac{1}{2}\int_t^s \beta_r^2 dr\bigg)\prod_{t<r\leq s}\big(1+\gamma_r(\Delta X_r)\big)e^{-\gamma_r(\Delta X_r)},$$

with $X_t = \int_0^t \int_U u\pi(du,ds)$.

Note that, classically, if $\gamma_t(e) \geq -1,\ d\mathbb{P}\otimes ds\otimes d\lambda(e)$-a.s., then $\Gamma_{t,.} \geq 0$ a.s. (see [16, Prop. 3.1]).

We make the following monotonicity assumption on f w.r.t. v

$(H6')$ For each $(y,z,v,v') \in \mathbb{R}\times\mathbb{R}^d\times(\mathcal{L}^2_\lambda)^2$, there exists a predictable process $\kappa = \kappa^{y,z,v,v'} : \Omega\times[0,T]\times U \to \mathbb{R}$ satisfying:

- $-1 \leq \kappa_t^{y,z,v,v'}(e)$;
- $|\kappa_t^{y,z,v,v'}(e)| \leq \vartheta(e)$, where $\vartheta \in \mathcal{L}^2_\lambda$ is such that

$$f(t,y,z,v) - f\big(t,y,z,v'\big) \leq \int_U \big(v(e)-v'(e)\big)\kappa_t^{y,z,v,v'}(e)\lambda(de),$$

$\mathbb{P}\otimes Leb\otimes\lambda$-a.e.

Notice that $(H6')$ implies $(H6)$ (see Section 5 in [9]).

We begin by showing that a linear GBSDE with jumps can be written as a conditional expectation via an exponential semimartingale. This result will be used to prove the comparison theorem.

Lemma 6. *Assume that $|\beta|$ is bounded and α is bounded from above. Suppose also that, $d\mathbb{P}\otimes dt\otimes\lambda(de)$-a.s.,*

$$-1 \leq \gamma_t(e), \tag{63}$$

and

$$\big|\gamma_t(e)\big| \leq \vartheta(e), \quad \textit{where } \vartheta \in \mathcal{L}^2_\lambda. \tag{64}$$

Let $(f_t)_{0\leq t\leq T}$ be a real-valued progressively measurable process, and let (Y,Z,V,M) be the solution of the linear GBSDE

$$\begin{aligned} Y_t = \xi &+ \int_t^T\Big[f_s + \alpha_s Y_s + \beta_s Z_s + \int_U \gamma_s(u)V_s(e)\lambda(de)\Big]ds + \int_t^T dR_s \\ &- \int_t^T\int_U V_s(e)\widehat{\pi}(de,ds) - \int_t^T Z_s dW_s - \int_t^T dM_s. \end{aligned} \tag{65}$$

Then, $\mathbb{E}\sup_{s\in[t,T]}|\Gamma_{t,s}|^p < +\infty$, and if

$$\mathbb{E}\bigg[|\xi|^p + \bigg(\int_0^T |f_s|ds\bigg)^p + |R|_T^p\bigg] < +\infty, \tag{66}$$

then the solution (Y,Z,V,M) belongs to Ξ^p.

Furthermore, the process (Y_t) *satisfies*

$$Y_t = \mathbb{E}\Big[\xi \Gamma_{t,T} + \int_t^T \Gamma_{t,s} f_s ds + \int_t^T \Gamma_{t,s-} dR_s \;\Big|\; \mathcal{F}_t\Big], \quad 0 \le t \le T, \ a.s. \tag{67}$$

Proof. We first show that $\mathbb{E}\sup_{s\in[t,T]} |\Gamma_{t,s}|^p < +\infty$. Indeed, by (63), as mentioned previously, it follows that $\Gamma_{t,\cdot} \ge 0$. Combining this with (64), using the fact that $|\beta|$ is bounded and α is bounded from above, and applying [16, Prop. A.1] yield that Γ is p-integrable, that is, $\mathbb{E}|\Gamma_{t,T}|^p < +\infty$. Hence, using Doob's inequality, we have that

$$\mathbb{E} \sup_{s\in[t,T]} |\Gamma_{t,s}|^p \le C_p \sup_{s\in[t,T]} \mathbb{E}|\Gamma_{t,s}|^p \le C_p \mathbb{E}|\Gamma_{t,T}|^p < +\infty, \tag{68}$$

as desired.

Next, let us show that (Y, Z, V, M) belongs to Ξ^p. Clearly, thanks to the assumptions made on α, β, and γ and the fact $(H6')$ implies $(H6)$, we can easily see that the generator of the linear GBSDE (65) satisfies assumptions $(H3)$, $(H5)$, and $(H6)$. Thus, in view of this and (66), applying Lemma (4) yields the claim.

It remains to show that (Y_t) satisfies (67). Indeed, by the Itô product formula we obtain

$$\begin{aligned} d(Y_s\Gamma_{t,s}) &= \Gamma_{t,s-} dY_s + Y_{s-} d\Gamma_{t,s} + d[\Gamma_{t,\cdot}, Y]_s \\ &= \Gamma_{t,s-}\Big(-f_s - \alpha_s Y_s - \beta_s Z_s - \int_U \gamma_s(u) V_s(u)\lambda(de)\Big)ds - \Gamma_{t,s-} dR_s \\ &\quad + \Gamma_{t,s-}\int_U V_s(e)\widehat{\pi}(de, ds) + \Gamma_{t,s-} Z_s dW_s + \Gamma_{t,s-} dM_s \\ &\quad + Y_{s-}\Gamma_{t,s-}\Big(\alpha_s ds + \beta_s dW_s + \int_U \gamma_s(u)\widehat{\pi}(de, ds)\Big) \\ &\quad + \Gamma_{t,s-}\beta_s Z_s ds + \Gamma_{t,s-}\int_U V_s(e)\gamma_s(u)\pi(de, ds) \\ &= -\Gamma_{t,s} f_s ds - \Gamma_{t,s-} dR_s + dN_s, \end{aligned}$$

with

$$\begin{aligned} dN_s &= \Gamma_{t,s-}\int_U \big(V_s(e) + Y_{s-}\gamma_s(e) + V_s(e)\gamma_s(e)\big)\widehat{\pi}(de, ds) \\ &\quad + \Gamma_{t,s-}(Z_s + Y_s\beta_s) dW_s + \Gamma_{t,s-} dM_s. \end{aligned}$$

Integrating between t and T yields

$$\xi\Gamma_{t,T} - Y_t = -\int_t^T \Gamma_{t,s} f_s ds - \int_t^T \Gamma_{t,s-} dR_s + \int_t^T dN_s \quad \text{a.s.} \tag{69}$$

In view of the boundedness assumptions made on the coefficients β and γ, combined with estimate (68) and the fact that $(Y, Z, V, M) \in \Xi^p$, it follows that the local martingale N is a uniformly integrable martingale. Therefore, taking the conditional expectation w.r.t. $\mathcal{F}_t$ in (69), we obtain

$$Y_t = \mathbb{E}\left[\xi \Gamma_{t,T} + \int_t^T \Gamma_{t,s} f_s ds + \int_t^T \Gamma_{t,s-} dR_s \,\Big|\, \mathcal{F}_t\right],$$

as desired. This ends the proof. □

Proposition 1. *We consider two sets of data* $(\xi_1, f_1 + dR_1)$ *and* $(\xi_2, f_2 + dR_2)$ *such that* ξ_1, ξ_2, R_1, *and* R_2 *satisfy* $(H1)$. *Moreover, we assume that* f_1 *and* f_2 *satisfy, respectively,* $(H1)$–$(H6)$ *and* $(H1)$–$(H5)$, $(H6')$. *Let* (Y^1, Z^1, V^1, M^1) *and* (Y^2, Z^2, V^2, M^2) *be respectively solutions of GBSDEs* (1) *associated with* $(\xi_1, f_1 + dR_1)$ *and* $(\xi_2, f_2 + dR_2)$ *in some space* Ξ^p *with* $p \geq 2$. *If* $\xi^1 \leq \xi^2$, $f_1(t, Y_t^1, Z_t^1, V_t^1) \leq f_2(t, Y_t^1, Z_t^1, V_t^1)$, *and for a.e.* t, $dR^1 \leq dR^2$, *then a.s.* $Y_t^1 \leq Y_t^2$ *for any* $t \in [0, T]$.

Proof. Put

$$\overline{Y} = Y^2 - Y^1, \quad \overline{Z} = Z^2 - Z^1, \quad \overline{V} = V^2 - V^1, \quad \overline{M} = M^2 - M^1, \quad \overline{R} = R^2 - R^1.$$

Then $(\overline{Y}, \overline{Z}, \overline{V}, \overline{M})$ satisfies

$$\overline{Y}_t = \overline{\xi} + \int_t^T h_s ds + \int_t^T d\overline{R}_s - \int_t^T \int_U \overline{\psi}_s(u) \widehat{\pi}(de, ds) - \int_t^T \overline{Z}_s dW_s - \int_t^T d\overline{M}_s,$$

where

$$h_s = f_2\big(Y_s^2, Z_s^2, \psi_s^2\big) - f_1\big(Y_s^1, Z_s^1, \psi_s^1\big).$$

Now we define

$$\begin{aligned}
f_s &= f_2\big(Y_s^1, Z_s^1, \psi_s^1\big) - f_1\big(Y_s^1, Z_s^1, \psi_s^1\big), \\
\alpha_s &= \frac{f_2(Y_s^2, Z_s^1, \psi_s^1) - f_2(Y_s^1, Z_s^1, \psi_s^1)}{\overline{Y}_s} \mathbb{1}_{\overline{Y}_s \neq 0}, \\
\beta_s &= \frac{f_2(Y_s^2, Z_s^2, \psi_s^1) - f_2(Y_s^2, Z_s^1, \psi_s^1)}{\overline{Z}_s} \mathbb{1}_{\overline{Z}_s \neq 0}.
\end{aligned}$$

Then

$$\begin{aligned}
h_s &= f_s + \alpha_s \overline{Y}_s + \beta_s \overline{Z}_s + f_2\big(Y_s^2, Z_s^2, V_s^2\big) - f_2\big(Y_s^2, Z_s^2, V_s^1\big) \\
&\geq f_s + \alpha_s \overline{Y}_s + \beta_s \overline{Z}_s + \int_U \kappa_s^{Y_s^2, Z_s^2, V_s^1, V_s^2} \overline{V}_s(u) \lambda(de),
\end{aligned} \tag{70}$$

since f_2 satisfies $(H6')$. Moreover, since f_2 is Lipschitz continuous w.r.t. z, $|\beta|$ is bounded by L, whereas, by Assumption $(H3)$, α is bounded from above. Moreover, the process $\kappa_s^{Y_s^2, Z_s^2, V_s^1, V_s^2}$ is controlled by $\vartheta \in \mathcal{L}_\lambda^2$. Note that, since $-1 \leq \kappa_t^{y,z,\psi,\phi}(e)$, it follows that $\Gamma_{t,.} \geq 0$ a.s. Furthermore, in view of the above, we have from Lemma 6 that $\Gamma_{t,.} \in \mathcal{S}^p$.

Now applying Itô's formula to $\bar{Y}_s \Gamma_{t,s}$ for $s \in [t, T]$ and then using inequality (70) together with the non negativity of Γ, we can derive, by doing the same computations as in the proof of Lemma 6, that

$$-d(\bar{Y}_s \Gamma_{t,s}) \geq \Gamma_{t,s} f_s ds + \Gamma_{t,s-} d\bar{R}_s - dN_s, \tag{71}$$

where N is a local martingale. Next, applying Lemma 5 yields that $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M})$ belongs to Ξ^p. Since $\Gamma_{t,.} \in \mathcal{S}^p$, this, combined with the boundedness of β and $\kappa_t^{y,z,\psi,\phi}(e)$, implies that N is in fact a martingale.

Therefore, integrating between t and T in (71) and then taking the conditional expectation w.r.t. $\mathcal{F}_t$, we deduce

$$\overline{Y}_t \geq \mathbb{E}\Big[\Gamma_{t,T}\overline{\xi} + \int_t^T \Gamma_{t,s} f_s ds + \int_t^T \Gamma_{t,s-} d\overline{R}_s \;\Big|\; \mathcal{F}_t\Big], \quad t \in [0, T], \text{ a.s.}$$

To conclude, recall that $\Gamma_{t,s} \geq 0$ a.s. and, by assumptions, $\overline{\xi} \geq 0$, $f_s \geq 0$, and $\int_t^T d\overline{R}_s \geq 0$. Consequently, it follows that, for all $t \in [0, T]$, $\overline{Y}_t \geq 0$ a.s. Since Y^1 and Y^2 are càdlàg processes, we obtain that $Y_t^1 \leq Y_t^2$ a.s., and the conclusion follows. □

Notice that assumptions $(H1)$–$(H6)$ made on f_1 are imposed only to ensure the existence of a solution (Y^1, Z^1, V^1, M^1). The following corollary, which follows immediately from Proposition 1, gives again a uniqueness result for GBSDE (1) in Ξ^p in dimension 1.

Corollary 1. *Let $p \geq 2$ and assume $(H1)$–$(H5)$ and $(H6')$. Then there exists at most one solution (Y, Z, V, M) to GBSDE (1) in Ξ^p.*

5 Generalized BSDEs with random terminal time

In this section, we study the issue of existence and uniqueness of $\mathbb{L}^p(p \geq 2)$-solutions of GBSDEs with random terminal time. We follow the approach in [14, Section 4]. Let τ be an $\mathcal{F}$-stopping time, not necessarily bounded. Assumptions considered in the case of GBSDE with constant time (precisely, $(H2)$, $(H3)$, $(H5)$, and $(H6)$) still hold except for $(H4)$ and $(H1)$, for which we give the analogues for $p \geq 2$:

$(H4')$ $\forall r > 0$, $\forall n \in \mathbb{N}$, the mapping $t \in [0, T] \to \sup_{|y| \leq r} |f(t, y, 0, 0) - f(t, 0, 0, 0)|$ belongs to $\mathbb{L}^1(\Omega \times (0, n))$.

$(H1')$ For some $\rho \in \mathbb{R}$ such that $\rho > \nu := \mu + \frac{2pL^2}{\alpha_p}$, where $\alpha_p = \min(\frac{p}{2}, p(p-1)3^{1-p})$,

$$\mathbb{E}\Big[e^{\rho p \tau}|\xi|^p + \Big(\int_0^\tau e^{\rho\tau}\big|f(s, 0, 0, 0)\big|ds\Big)^p + \Big(\int_0^\tau e^{\rho\tau} d|R|_s\Big)^p\Big] < \infty. \tag{72}$$

Finally, we will need the following additional assumption on ξ and f:
$(H7)$ ξ is $\mathcal{F}_\tau$-measurable, and $\mathbb{E}[(\int_0^\tau e^{\rho\tau}|f(t,\xi_t,\eta_t,\gamma_t)|ds)^p] < +\infty$, where $\xi_t = \mathbb{E}(\xi|\mathcal{F}_t)$ and (η, γ, N) are given by the martingale representation

$$\xi = \mathbb{E}(\xi) + \int_0^{+\infty} \eta_s dW_s + \int_0^{+\infty}\int_U \gamma_s(e)\widehat{\pi}(de,ds) + N_\tau,$$

with N orthogonal to W and $\widetilde{\pi}$. Moreover, the following holds:

$$\mathbb{E}\Big[\Big(\int_0^{+\infty} |\eta_s|^2 ds\Big)^{\frac{p}{2}} + \Big(\int_0^{+\infty}\int_U |\gamma_s(e)|^2 \lambda(de)ds\Big)^{\frac{p}{2}} + [N]_\tau^{\frac{p}{2}}\Big] < +\infty.$$

Next, let us make precise the notion of a solution of GBSDE with random terminal time.

Definition 2. We say that a quadruple $(Y, Z, V, M) \in \mathcal{S} \times \mathcal{H}(0,T) \times \mathcal{P} \times \mathbb{M}_{loc}$ with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k \times \mathbb{R}^k$ is a solution of GBSDE (1) with random terminal time τ and data $(\xi, f + dR)$ if

- on $\{t \geq \tau\}$, $Y_t = \xi$ and $Z_t = V_t = M_t = 0$, $\mathbb{P}$-a.s.,
- $t \to f(t, Y_t, Z_t, V_t)\mathbb{1}_{\{t\leq\tau\}} \in L^1(0,+\infty)$, $Z \in \mathbb{L}^2_{loc}(W)$, $V \in G_{loc}(\pi)$, and
- $\mathbb{P}$-a.s., for all $t \in [0, T]$,

$$\begin{aligned} Y_{t\wedge\tau} = Y_{T\wedge\tau} &+ \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, V_s)ds + \int_{t\wedge\tau}^{T\wedge\tau} dR_s - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s \\ &- \int_{t\wedge\tau}^{T\wedge\tau}\int_U V_s(e)\widehat{\pi}(de,ds) - \int_{t\wedge\tau}^{T\wedge\tau} dM_s. \end{aligned} \tag{73}$$

Furthermore, a solution is said to be $\mathbb{L}^p$ if we have

$$\begin{aligned} \mathbb{E}\Big[e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p &+ \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^p ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}|Z_s|^2 ds \\ &+ \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}\|V_s\|^2_{L^2_\lambda} ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2} d[M]_s \Big] \\ &< +\infty. \end{aligned} \tag{74}$$

Proposition 2. *Assume* $(H1')$, $(H2)$, $(H3)$, $(H4')$, $(H5)$, $(H6)$, *and* $(H7)$. *Then, there exists at most one solution to GBSDE* (1) *that satisfies estimate* (74).

Proof. Assume that there exist two solutions (Y, Z, V, M) and (Y', Z', V', M') of GBSDE (73) that satisfy estimate (74). Set $(\bar{Y}, \bar{Z}, \bar{V}, \bar{M}) = (Y - Y', Z - Z', V - V', M - M')$.

Applying Itô's formula, as in step 2 of Proposition 4, to $e^{p\rho s}|\bar{Y}_s|^p$ over the interval $[t\wedge\tau, T\wedge\tau]$, we obtain an analogue of (41)

$$\begin{aligned}
&e^{p\rho(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p\\
&\quad= e^{p\rho(T\wedge\tau)}|\bar{Y}_{T\wedge\tau}|^p\\
&\qquad+ p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big[\bar{Y}_s|\bar{Y}_s|^{p-2}\big(f(s,Y_s,Z_s,V_s)-f\big(s,Y'_s,Z'_s,V'_s\big)\big)-\rho|\bar{Y}_s|^p\big]ds\\
&\qquad- p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}d\bar{M}_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\bar{Y}_s|\bar{Y}_s|^{p-2}\bar{Z}_s dW_s\\
&\qquad- p\int_{t\wedge\tau}^{T\wedge\tau}\int_U e^{p\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}\bar{V}_s(e)\widehat{\pi}(de,ds)\\
&\qquad- \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\,\mathrm{Trace}\big(D^2\theta(\bar{Y}_s)\bar{Z}_s\bar{Z}_s^t\big)ds\\
&\qquad- \int_{t\wedge\tau}^{T\wedge\tau}\int_U e^{p\rho s}\big(\big|\bar{Y}_{s-}+\bar{Y}_s(e)\big|^p\\
&\qquad- |\bar{Y}_{s-}|^p - p\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}\bar{Y}_s(e)\big)\pi(de,ds) - \aleph_t,
\end{aligned}\tag{75}$$

where

$$\begin{aligned}
\aleph_t &= \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\sum_{1\le i,j\le d}\frac{\partial^2\theta}{\partial y_i\partial y_j}(Y_s)d\big[\bar{M}^i,\bar{M}^j\big]_s^c\\
&\quad+ \sum_{t\wedge\tau<s\le T\wedge\tau} e^{p\rho s}\big(|\bar{Y}_{s-}+\Delta\bar{M}_s|^p - |\bar{Y}_{s-}|^p - pY_{s-}|\bar{Y}_{s-}|^{p-2}\Delta\bar{M}_s\big).
\end{aligned}\tag{76}$$

The following estimates, which are analogues of (43) and (44), hold:

$$\aleph_t \ge \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\bar{Y}_{s-}|d[\bar{M}]_s,\tag{77}$$

and

$$\begin{aligned}
&-\int_{t\wedge\tau}^{T\wedge\tau}\int_U e^{p\rho s}\big(\big|\bar{Y}_{s-}+\bar{V}_s(e)\big|^p - |\bar{Y}_{s-}|^p - p\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}\bar{Y}_s(e)\big)\pi(de,ds)\\
&\quad\le -p(p-1)3^{1-p}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\bar{Y}_{s-}|^{p-2}\big|\bar{Y}_s(e)\big|^2\pi(de,ds),
\end{aligned}\tag{78}$$

where $\alpha_p = \min(\frac{p}{2}, p(p-1)3^{1-p})$.

Therefore, rearranging (75), in view of (42), (77), and (78) yields

$$\begin{aligned}
&e^{p\rho(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\bar{Y}_s|^{p-2}|\bar{Z}_s|^2 ds\\
&\qquad+ \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\int_U |\bar{Y}_s|^{p-2}\big|\bar{V}_s(e)\big|^2\pi(de,ds) + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|\bar{Y}_{s-}|d[\bar{M}]_s
\end{aligned}$$

$$
\begin{aligned}
&\leq e^{p\rho(T\wedge\tau)}|\bar{Y}_{T\wedge\tau}|^p \\
&\quad + p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\big[\bar{Y}_s|\bar{Y}_s|^{p-2}\big(f(s,Y_s,Z_s,V_s) - f\big(s,Y'_s,Z'_s,V'_s\big)\big) - \rho|\bar{Y}_s|^p\big]ds \\
&\quad - p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}d\bar{M}_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\bar{Y}_s|\bar{Y}_s|^{p-2}\bar{Z}_s dW_s \\
&\quad - p\int_{t\wedge\tau}^{T\wedge\tau}\int_U e^{\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}\bar{V}_s(e)\widehat{\pi}(de,ds).
\end{aligned}
\tag{79}
$$

But from the assumptions on f (using (46) with $\epsilon = \frac{\alpha_p}{p}$) and Young's inequality we have that

$$
\begin{aligned}
&\bar{Y}_s|\bar{Y}_s|^{p-2}\big(f(s,Y_s,Z_s,V_s) - f\big(s,Y'_s,Z'_s,V'_s\big)\big) - \rho|\bar{Y}_s|^p \\
&\quad \leq \left(\mu + \frac{2pL^2}{\alpha_p} - \rho\right)|\bar{Y}_s|^p + \frac{\alpha_p}{p}|\bar{Y}_s|^{p-2}|\bar{Z}_s|^2 + \frac{\alpha_p}{p}|\bar{Y}_s|^{p-2}\|\bar{V}_s\|^2_{\mathbb{L}^2_\lambda} \\
&\quad \leq \frac{\alpha_p}{p}|\bar{Y}_s|^{p-2}|\bar{Z}_s|^2 + \frac{\alpha_p}{p}|\bar{Y}_s|^{p-2}\|\bar{V}_s\|^2_{\mathbb{L}^2_\lambda}.
\end{aligned}
\tag{80}
$$

Furthermore, observe that by the integrability conditions on the solution all the local martingales appearing in (79) are uniformly integrable. Moreover, the following holds:

$$
\begin{aligned}
&\mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\int_U |Y_s|^{p-2}\big|V_s(e)\big|^2\pi(de,ds) \\
&\quad = \mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\int_U |Y_s|^{p-2}\big|V_s(e)\big|^2\lambda(de)ds.
\end{aligned}
\tag{81}
$$

Thus, taking the expectation in (79), we obtain, in view of the above, that

$$
\mathbb{E}e^{p\rho(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p \leq \mathbb{E}e^{p\rho(T\wedge\tau)}|\bar{Y}_{T\wedge\tau}|^p. \tag{82}
$$

Note that the same result holds with ρ replaced by ρ', with $\mu + \frac{p^2L^2}{\alpha_p^2} < \rho' < \rho$. Therefore, we have, for any $0 \leq t \leq T$,

$$
\mathbb{E}e^{p\rho'(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p \leq e^{p(\rho'-\rho)T}\mathbb{E}e^{p\rho(T\wedge\tau)}|\bar{Y}_{T\wedge\tau}|^p. \tag{83}
$$

Consequently, letting $T \to +\infty$, we deduce in view of estimate (74) that $\bar{Y}_t = 0$.

Since (Y, Z, V, M) and (Y', Z', V', M') satisfy GBSDE (73) with $Y = Y'$, then by the uniqueness of the Doob–Meyer decomposition of semimartingales it follows that $(Z, V, M) = (Z', V', M')$, whence the uniqueness of the solution of (73). □

Proposition 3. *Assume that* $(H1')$, $(H2)$, $(H3)$, $(H4')$, $(H5)$, $(H6)$, *and* $(H7)$ *are in force. Then, GBSDE* (1) *has a solution satisfying*

$$
\mathbb{E}\Big[\sup_{t\geq 0} e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p + e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^p ds
$$

$$+\int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}|Z_s|^2 ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}\|V_s\|^2_{L^2_\lambda} ds$$
$$+\int_0^{T\wedge\tau} e^{p\rho s}|Y_{s-}|^{p-2} d[M]_s\Bigg]$$
$$\leq C\mathbb{E}\Bigg[e^{p\rho\tau}|\xi|^p + \left(\int_0^\tau e^{\rho\tau}\big|f(s,0,0,0)\big|ds\right)^p + \left(\int_0^\tau e^{\rho\tau} d|R|_s\right)^p\Bigg]. \quad (84)$$

Moreover,

$$\mathbb{E}\Bigg[\left(\int_0^\tau e^{2\rho s}|Z_s|^2 ds\right)^{\frac{p}{2}} + \left(\int_0^\tau\int_U e^{2\rho s}\big|V_s(e)\big|^2\lambda(de)ds\right)^{\frac{p}{2}}$$
$$+\left(\int_0^\tau e^{2\rho s} d[M]_s\right)^{\frac{p}{2}}\Bigg]$$
$$\leq C\mathbb{E}\Bigg[e^{p\rho\tau}|\xi|^p + \left(\int_0^\tau e^{\rho\tau}\big|f(s,0,0,0)\big|ds\right)^p + \left(\int_0^\tau e^{\rho\tau} d|R|_s\right)^p\Bigg], \quad (85)$$

for some constant $C > 0$ depending only on p, L, and μ.

Proof. We follow the line of the argument of [14, Thm. 4.1]. For each $n \in \mathbb{N}$, we construct a solution $\{(Y^n, Z^n, V^n, M^n)\}$ as follows. By Theorem 2, on the interval $[0, n]$,

$$Y_t^n = \mathbb{E}(\xi \mid \mathcal{F}_n) + \int_t^n \mathbb{1}_{[0,\tau]}(s) f\big(s, Y_s^n, Z_s^n, V_s^n\big)ds + \int_t^n \mathbb{1}_{[0,\tau]}(s) dR_s$$
$$-\int_t^n Z_s^n dW_s - \int_t^n\int_U V_s^n(e)\widehat{\pi}(de, ds) - \int_t^n dM_s^n, \quad (86)$$

and for $t \geq n$, we have by assumption $(H7)$ that $Y_t^n = \xi_t$, $Z_t^n = \eta_t$, $V_t^n(e) = \gamma_t(e)$, $M_t^n = N_t$.

Step 1. We first show that (Y^n, Z^n, V^n, M^n) satisfies estimate (84). Applying Itô's formula to $e^{p\rho s}|Y_s^n|^p$ over the interval $[t\wedge\tau, T\wedge\tau]$ for $0 \leq t \leq T \leq n$ and combining with (42), (77), and (78) yield

$$e^{p\rho(t\wedge\tau)}\big|Y_{t\wedge\tau}^n\big|^p + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y_s^n\big|^{p-2}\big|Z_s^n\big|^2 ds$$
$$+\alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\int_U \big|Y_s^n\big|^{p-2}\big|V_s^n(e)\big|^2\pi(de, ds) + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y_{s-}^n\big|d\big[M^n\big]_s$$
$$\leq e^{p\rho(T\wedge\tau)}\big|Y_{T\wedge\tau}^n\big|^p + p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\big[Y_s^n\big|Y_s^n\big|^{p-2} f\big(s, Y_s^n, Z_s^n, V_s^n\big) - \rho\big|Y_s^n\big|^p\big]ds$$
$$+p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s} Y_s^n\big|Y_s^n\big|^{p-2} dR_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s} Y_{s-}^n\big|Y_{s-}^n\big|^{p-2} dM_s^n$$
$$-p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s} Y_s^n\big|Y_s^n\big|^{p-2} Z_s^n dW_s$$
$$-p\int_{t\wedge\tau}^{T\wedge\tau}\int_U e^{\rho s} Y_{s-}^n\big|Y_{s-}^n\big|^{p-2} V_s^n(e)\,\widehat{\pi}(de, ds).$$

But, from the assumptions on f combined with Young's inequality we get, for a small enough constant $\delta > 0$,

$$y|y|^{p-2}f(t,y,z,v) \leq \left(\mu + \frac{2pL^2}{\alpha_p - p\delta}\right)|y|^p + |y|^{p-1}\big|f(t,0,0,0)\big| \\ + \left(\frac{\alpha_p}{p} - \delta\right)|y|^{p-2}|z|^2 + \left(\frac{\alpha_p}{p} - \delta\right)|y|^{p-2}\|v\|_{\mathbb{L}^2_\lambda}.$$

Then, choosing δ such that $\mu + \frac{2pL^2}{\alpha_p - p\delta} < \rho$, we deduce from the above that

$$\begin{aligned}
& e^{p\alpha(t\wedge\tau)}\big|Y^n_{t\wedge\tau}\big|^p + p\bar{\rho}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\alpha s}\big|Y^n_s\big|^p ds + p\delta\int_{t\wedge\tau}^{T\wedge\tau} e^{p\alpha s}\big|Y^n_s\big|^{p-2}\big|Z^n_s\big|^2 ds \\
&\quad + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\alpha s}\big|Y^n_s\big|^{p-2}\big|V^n_s(e)\big|^2\pi(de,ds) + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\alpha s}\big|Y^n_s\big|^{p-2}d\big[M^n\big]_s \\
&\quad - p\left(\frac{\alpha_p}{p} - \delta\right)\int_{t\wedge\tau}^{T\wedge\tau} e^{p\alpha s}\big|Y^n_s\big|^{p-2}\big\|V^n_s\big\|^2_{\mathcal{L}^2_\lambda} ds \\
&\leq e^{p\alpha(T\wedge\tau)}\big|Y^n_{T\wedge\tau}\big|^p + p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\alpha s}\big|Y^n_s\big|^{p-1}\big|f(s,0,0,0)\big|ds \\
&\quad + p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\big|Y^n_s\big|^{p-1}d|R|_s - p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}Y^n_{s-}\big|Y^n_{s-}\big|^{p-2}dM^n_s \\
&\quad - p\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}Y^n_s\big|Y^n_s\big|^{p-2}Z^n_s dW_s \\
&\quad - p\int_{t\wedge\tau}^{T\wedge\tau}\int_U e^{\rho s}Y^n_{s-}\big|Y^n_{s-}\big|^{p-2}V^n_s(e)\,\widehat{\pi}(de,ds).
\end{aligned} \tag{87}$$

Note that all the local martingales in the last inequality are true martingales. Thus, taking the expectation in (87), we get in view of (81) that

$$\begin{aligned}
&\mathbb{E}\Bigg[e^{p\rho(t\wedge\tau)}\big|Y^n_{t\wedge\tau}\big|^p + p\delta\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y^n_s\big|^p ds + p\delta\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y^n_s\big|^{p-2}\big|Z^n_s\big|^2 ds \\
&\quad + p\delta\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y^n_s\big|^{p-2}\big\|V^n_s\big\|_{\mathcal{L}^2_\lambda} ds + \alpha_p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y^n_s\big|^{p-2}d\big[M^n\big]_s\Bigg] \\
&\leq \mathbb{E}[X],
\end{aligned} \tag{88}$$

where

$$X = e^{p\rho(T\wedge\tau)}\big|Y^n_{T\wedge\tau}\big|^p + p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y^n_s\big|^{p-1}\big|f(s,0,0,0)\big|ds \\ + p\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}\big|Y^n_{s-}\big|^{p-1}d|R|_s.$$

Next, as in the proof of Lemma 4, including a $\sup_{s\in[t,T]}$ in (87) and applying the BDG inequality, we get in view of (88) that

$$\mathbb{E}\Big[\sup_{s\in[t,T]} e^{p\rho(s\wedge\tau)}\big|Y^n_{s\wedge\tau}\big|^p\Big] \leq C_p\mathbb{E}[X]. \tag{89}$$

But by Young's inequality we have

$$\begin{aligned} & p\mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-1}\big|f(s,0,0,0)\big|ds \\ & \quad \le p\mathbb{E}\Big(\sup_{s\in[t\wedge\tau,T\wedge\tau]} e^{(p-1)\rho s}|Y_t|^{p-1}\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\big|f(s,0,0,0)\big|ds\Big) \\ & \quad \le \frac{1}{6}\mathbb{E}\Big(\sup_{s\in[t\wedge\tau,T\wedge\tau]} e^{p\rho s}|Y_t|^{p}\Big) + d'_p\mathbb{E}\Big(\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}\big|f(s,0,0,0)\big|ds\Big)^p, \end{aligned} \tag{90}$$

and

$$\begin{aligned} & p\mathbb{E}\int_{t\wedge\tau}^{T\wedge\tau} e^{p\rho s}|Y_{s-}|^{p-1}d|R|_s \\ & \quad \le \frac{1}{6}E\big(\sup_{s\in[t\wedge\tau,T\wedge\tau]} e^{p\rho s}|Y_t|^{p}\big) + d''_p\mathbb{E}\Big(\int_{t\wedge\tau}^{T\wedge\tau} e^{\rho s}d|R|_s\Big)^p. \end{aligned} \tag{91}$$

Consequently, combining (89) with (90) and (91) and letting $T \to +\infty$, we deduce that

$$\begin{aligned} & \mathbb{E}\Big[\sup_{t\ge 0} e^{p\rho(t\wedge\tau)}\big|Y^n_{t\wedge\tau}\big|^p\Big] \\ & \quad \le C''_p\mathbb{E}\Big[e^{p\rho\tau}|\xi|^p + \Big(\int_0^{\tau} e^{\rho s}\big|f(s,0,0,0)\big|ds\Big)^p + \Big(\int_0^{\tau} e^{\rho s}d|R|_s\Big)^p\Big]. \end{aligned} \tag{92}$$

Finally, going back to (88), we conclude in view of (90), (91), and (92) that estimate (84) holds for (Y^n, Z^n, V^n, M^n).

Step 2. Let us show that (Y^n, Z^n, V^n, M^n) is a Cauchy sequence. For $m > n$, define

$$\bar{Y}_t = Y^m_t - Y^n_t, \quad \bar{Z}_t = Z^m_t - Z^n_t, \quad \bar{V}_t = V^m_t - V^n_t, \quad \bar{M}_t = M^m_t - M^n_t.$$

For $n \le t \le m$, we have

$$\begin{aligned} \bar{Y}_t = & \int_{t\wedge\tau}^{m\wedge\tau} f\big(s, Y^m_s, Z^m_s, V^m_s\big)ds - \int_{t\wedge\tau}^{m\wedge\tau} \bar{Z}_s dW_s \\ & - \int_{t\wedge\tau}^{m\wedge\tau}\int_U \bar{V}_s(e)\widehat{\pi}(du, ds) - \bar{M}_{m\wedge\tau} + \bar{M}_{t\wedge\tau}. \end{aligned}$$

Consequently, again for $n \le t \le m$,

$$\begin{aligned} & e^{p\rho(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p + \alpha_p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\bar{Y}_s|^{p-2}|\bar{Z}_s|^2 ds \\ & \quad + \alpha_p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}\int_U |\bar{Y}_s|^{p-2}\big|\bar{V}_s(e)\big|^2\pi(de, ds) + \alpha_p\int_{t\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\bar{Y}_{s-}|d[\bar{M}]_s \\ & \quad \le p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\big[\bar{Y}_s|\bar{Y}_s|^{p-2}f\big(s, Y^m_s, Z^m_s, V^m_s\big) - \rho|\bar{Y}_s|^p\big]ds \\ & \quad - p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\bar{Y}_s|\bar{Y}_s|^{p-2}\bar{Z}_s dW_s - p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}d\bar{M}_s \end{aligned}$$

$$
\begin{aligned}
&- p\int_{t\wedge\tau}^{m\wedge\tau}\int_U e^{\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}\bar{V}_s(e)\,\widehat{\pi}(de,ds)\\
&\leq p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\big[\mu|\bar{Y}_s|^p + L|\bar{Y}_s|^{p-1}|\bar{Z}_s| + L|\bar{Y}_s|^{p-1}\big\|\bar{V}_s(e)\big\|_{\mathcal{L}^2_\lambda} - \rho|\bar{Y}_s|^p\big]ds\\
&\quad + p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\bar{Y}_s|\bar{Y}_s|^{p-2}f(s,\xi_s,\eta_s,\gamma_s)ds - p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\bar{Y}_s|\bar{Y}_s|^{p-2}\bar{Z}_s dW_s\\
&\quad - p\int_{t\wedge\tau}^{m\wedge\tau} e^{\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}d\bar{M}_s\\
&\quad - p\int_{t\wedge\tau}^{m\wedge\tau}\int_U e^{\rho s}\bar{Y}_{s-}|\bar{Y}_{s-}|^{p-2}\bar{V}_s(e)\,\widehat{\pi}(de,ds).
\end{aligned}
$$

We deduce by already used arguments that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{n\leq t\leq m} e^{p\rho(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p + \int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\bar{Y}_s|^p ds + \int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\bar{Y}_s|^{p-2}|\bar{Z}_s|^2 ds\\
&\quad + \int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}\int_U |\bar{Y}_s|^{p-2}\big\|\bar{V}_s(e)\big\|^2_{\mathcal{L}^2_\lambda}ds + \int_{n\wedge\tau}^{m\wedge\tau} e^{p\rho s}|\bar{Y}_{s-}|d[\bar{M}]_s\Big]\\
&\quad \leq C\mathbb{E}\Big(\int_{n\wedge\tau}^{\tau} e^{\rho s}\big|f(s,\xi_s,\eta_s,\gamma_s)\big|ds\Big)^p,
\end{aligned}
$$

and the last term tends to zero as $n \to \infty$.

Next, for $t \leq n$,

$$
\begin{aligned}
\bar{Y}_t &= \bar{Y}_n + \int_{t\wedge\tau}^{n\wedge\tau}\big(f\big(s,Y_s^m,Z_s^m,V_s^m\big) - f\big(s,Y_s^n,Z_s^n,V_s^n\big)\big)ds - \int_{t\wedge\tau}^{n\wedge\tau}\bar{Z}_s dW_s\\
&\quad - \int_{t\wedge\tau}^{n\wedge\tau}\int_U \widehat{V}_s(e)\widehat{\pi}(de,ds) - \bar{M}_{n\wedge\tau} + \bar{M}_{t\wedge\tau}.
\end{aligned}
$$

Arguing as in Proposition 2, we get

$$
\begin{aligned}
\mathbb{E}e^{p\rho(t\wedge\tau)}|\bar{Y}_{t\wedge\tau}|^p\mathbb{E}\int_0^\tau e^{p\rho s}|\bar{Y}_s|^p ds &\leq \mathbb{E}e^{p\rho(n\wedge\tau)}|\bar{Y}_n|^p\\
&\leq C\mathbb{E}\Big(\int_{n\wedge\tau}^{\tau} e^{\rho s}\big|f(s,\xi_s,\eta_s,\gamma_s)\big|ds\Big)^p,
\end{aligned}
$$

and letting $n \to \infty$, the convergence of the sequence Y^n follows.

Next, it remains to show the convergence of the martingale part (Z^n, V^n, M^n). We follow the proof of Lemma 4. We apply Itô's formula to $e^{2\rho s}|\bar{Y}_s|^2$ for $n \leq t \leq m$:

$$
\begin{aligned}
&\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}|\bar{Z}_s|^2 ds + \int_{t\wedge\tau}^{m\wedge\tau}\int_U e^{2\rho s}\big|\bar{V}_s(e)\big|^2\pi(de,ds) + \int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}d[\bar{M}]_s\\
&\quad = 2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\big[\bar{Y}_s\big(f\big(s,Y_s^m,Z_s^m,V_s^m\big) - f(s,\xi_s,\eta_s,\gamma_s)\big) - \rho|\bar{Y}_s|^2\big]ds\\
&\qquad + 2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\bar{Y}_s f(s,\xi_s,\eta_s,\gamma_s)ds - 2\int_0^{\tau_n} e^{2\rho s}\bar{Y}_s\bar{Z}_s dW_s\\
&\qquad - 2\int_{t\wedge\tau}^{m\wedge\tau}\int_U e^{2\rho s}\bar{Y}_{s-}\bar{V}_s(e)\widehat{\pi}(de,ds) - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{2\rho s}\bar{Y}_{s-}d\bar{M}_s.
\end{aligned}
$$

Mimicking the same argumentation used to obtain (34) (assumptions on f and BDG and Young inequalities) leads to

$$\begin{aligned}&\mathbb{E}\bigg[\bigg(\int_{n\wedge\tau}^{m\wedge\tau} e^{2\rho s}|\bar{Z}_s|^2 ds\bigg)^{\frac{p}{2}} + \bigg(\int_{n\wedge\tau}^{m\wedge\tau}\int_U e^{2\rho s}\big|\bar{V}_s(e)\big|^2\lambda(de)ds\bigg)^{\frac{p}{2}} \\ &\quad + \bigg(\int_{n\wedge\tau}^{m\wedge\tau} e^{2\rho s} d[\bar{M}]_s\bigg)^{\frac{p}{2}}\bigg] \\ &\le C(p,T)\bigg[\mathbb{E}\sup_{t\ge n} e^{p\rho s}|\bar{Y}_t|^p + \mathbb{E}\bigg(\int_{n\wedge\tau}^{\tau} e^{\rho s}\big|f(s,\xi_s,\eta_s,\gamma_s)\big|ds\bigg)^p\bigg]. \end{aligned} \tag{93}$$

Next, arguing similarly for the case $t \le n$, we get

$$\begin{aligned}&\mathbb{E}\bigg[\bigg(\int_0^{n\wedge\tau} e^{2\rho s}|\bar{Z}_s|^2 ds\bigg)^{\frac{p}{2}} + \bigg(\int_0^{n\wedge\tau}\int_U e^{2\rho s}\big|\bar{V}_s(e)\big|^2\lambda(de)ds\bigg)^{\frac{p}{2}} \\ &\quad + \bigg(\int_0^{n\wedge\tau} e^{2\rho s} d[\bar{M}]_s\bigg)^{\frac{p}{2}}\bigg] \\ &\le C(p,T)\big[\mathbb{E}\sup_{t\ge n} e^{p\rho n\wedge\tau}|\bar{Y}_{n\wedge\tau}|^p\big]. \end{aligned} \tag{94}$$

Consequently, letting $n \to \infty$, we deduce from the above that, in both cases, the sequence (Z^n, V^n, M^n) is a Cauchy sequence for the norm

$$\mathbb{E}\bigg(\int_0^{\tau} e^{2\rho s}|\bar{Z}_s|^2 ds\bigg)^{\frac{p}{2}} + \mathbb{E}\bigg(\int_0^{\tau}\int_U e^{2\rho s}\big|\bar{V}_s(e)\big|^2\lambda(de)ds\bigg)^{\frac{p}{2}} + \mathbb{E}\bigg(\int_0^{\tau} e^{2\rho s} d[\bar{M}]_s\bigg)^{\frac{p}{2}},$$

and thus it converges to (Z, V, M). Finally, the limit (Y, Z, V, M) is a solution of GBSDE (73) that satisfies estimates (84) and (85). □

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