

# Randomly stopped maximum and maximum of sums with consistently varying distributions

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**Abstract** Let  $\{\xi_1, \xi_2, \dots\}$  be a sequence of independent random variables, and  $\eta$  be a counting random variable independent of this sequence. In addition, let  $S_0 := 0$  and  $S_n := \xi_1 + \xi_2 + \dots + \xi_n$  for  $n \geq 1$ . We consider conditions for random variables  $\{\xi_1, \xi_2, \dots\}$  and  $\eta$  under which the distribution functions of the random maximum  $\xi_{(\eta)} := \max\{0, \xi_1, \xi_2, \dots, \xi_\eta\}$  and of the random maximum of sums  $S_{(\eta)} := \max\{S_0, S_1, S_2, \dots, S_\eta\}$  belong to the class of consistently varying distributions. In our consideration the random variables  $\{\xi_1, \xi_2, \dots\}$  are not necessarily identically distributed.

**Keywords** Heavy tail, consistently varying tail, randomly stopped maximum, randomly stopped maximum of sums, closure property

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## 1 Introduction

Let  $\{\xi_1, \xi_2, \dots\}$  be a sequence of independent random variables (r.v.s) with distribution functions (d.f.s)  $\{F_{\xi_1}, F_{\xi_2}, \dots\}$ , and let  $\eta$  be a counting r.v., that is, an integer-valued, nonnegative, and nondegenerate at zero r.v. In addition, suppose that the r.v.  $\eta$  and r.v.s  $\{\xi_1, \xi_2, \dots\}$  are independent.

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Let  $S_0 = 0$ ,  $S_n = \xi_1 + \xi_2 + \dots + \xi_n$  for  $n \in \mathbb{N}$ , and let

$$S_\eta = \sum_{k=1}^{\eta} \xi_k$$

be the randomly stopped sum of r.v.s  $\{\xi_1, \xi_2, \dots\}$ .

Similarly, let  $\xi_{(\eta)} = \max\{0, \xi_1, \xi_2, \dots, \xi_n\}$  for  $n \in \mathbb{N}$ ,  $\xi_{(0)} = 0$ , and let

$$\xi_{(\eta)} = \begin{cases} \max\{0, \xi_1, \xi_2, \dots, \xi_\eta\} & \text{if } \eta \geq 1, \\ 0 & \text{if } \eta = 0 \end{cases}$$

be the randomly stopped maximum of r.v.s  $\{\xi_1, \xi_2, \dots\}$ .

Finally, let  $S_{(\eta)} := \max\{S_0, S_1, S_2, \dots, S_n\}$  for  $n \geq 0$ , and let

$$S_{(\eta)} := \max\{S_0, S_1, S_2, \dots, S_\eta\}$$

be the randomly stopped maximum of sums  $S_0, S_1, S_2, \dots$ .

We denote the distribution functions (d.f.s) of  $S_\eta$ ,  $\xi_{(\eta)}$ , and  $S_{(\eta)}$  by  $F_{S_\eta}$ ,  $F_{\xi_{(\eta)}}$ , and  $F_{S_{(\eta)}}$  respectively, together with their tails  $\overline{F}_{S_\eta}$ ,  $\overline{F}_{\xi_{(\eta)}}$ , and  $\overline{F}_{S_{(\eta)}}$ . For any positive  $x$ , we have

$$\begin{aligned} \overline{F}_{S_\eta}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \mathbb{P}(S_n > x), \\ \overline{F}_{\xi_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \mathbb{P}(\xi_{(n)} > x), \\ \overline{F}_{S_{(\eta)}}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\eta = n) \mathbb{P}(S_{(n)} > x). \end{aligned}$$

In [14], conditions were found for the d.f.s  $F_{S_\eta}$  to belong to the class of consistently varying distributions. In this paper, we are interested in sufficient conditions under which the d.f.s  $F_{\xi_{(\eta)}}$  and  $F_{S_{(\eta)}}$  have consistently varying tails.

Throughout this paper, for two vanishing (at infinity) functions  $f$  and  $g$ ,  $f(x) = o(g(x))$  means that  $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$ , and  $f(x) \sim g(x)$  means that  $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ . Also, we denote the support of a counting r.v.  $\eta$  by

$$\text{supp}(\eta) := \{n \in \mathbb{Z}_+ : \mathbb{P}(\eta = n) > 0\}.$$

Before discussing the properties of  $F_{\xi_{(\eta)}}$  and  $F_{S_{(\eta)}}$ , we recall the definitions of some classes of heavy-tailed d.f.s. For a d.f.  $F$ , we denote  $\overline{F}(x) = 1 - F(x)$  for real  $x$ .

- A d.f.  $F$  is heavy-tailed ( $F \in \mathcal{H}$ ) if for every  $\delta > 0$ ,

$$\lim_{x \rightarrow \infty} \overline{F}(x) e^{\delta x} = \infty.$$

- A d.f.  $F$  is long-tailed ( $F \in \mathcal{L}$ ) if for every  $y$  (equivalently, for some  $y > 0$ ),  $\overline{F}(x + y) \sim \overline{F}(x)$ .
- A d.f.  $F$  has dominantly varying tail ( $F \in \mathcal{D}$ ) if for every  $y \in (0, 1)$  (equivalently, for some  $y \in (0, 1)$ ),

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

- A d.f.  $F$  has consistently varying tail ( $F \in \mathcal{C}$ ) if

$$\lim_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

- A d.f.  $F$  has regularly varying tail ( $F \in \mathcal{R}$ ) if there is  $\alpha \geq 0$  such that

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha},$$

for all  $y > 0$ .

- A d.f.  $F$  supported on the interval  $[0, \infty)$  is subexponential ( $F \in \mathcal{S}$ ) if

$$\lim_{x \rightarrow \infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2. \tag{1}$$

If a d.f.  $G$  is supported on  $\mathbb{R}$ , then we say that  $G$  is subexponential ( $G \in \mathcal{S}$ ) if the d.f.  $F(x) = G(x)\mathbb{1}_{[0, \infty)}(x)$  satisfies relation (1).

It is known (see, e.g., [5, 12, 15], and Chapters 1.4 and A3 in [10]) that these classes satisfy the following inclusions:

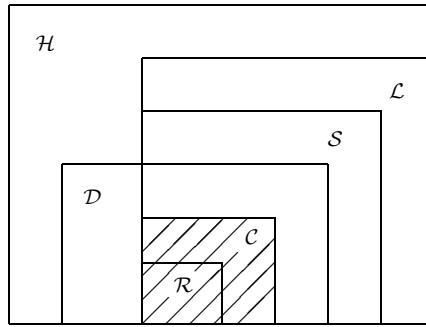
$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \quad \mathcal{D} \subset \mathcal{H}.$$

These inclusions are depicted in Fig. 1 borrowed from the paper [14]. In this figure, the class  $\mathcal{C}$  of distributions having consistently varying tails is highlighted.

It should be noted that the subject of the paper is partially motivated by the closure problem of the random convolution. In the case of independent and identically distributed (i.i.d.) r.v.s  $\{\xi, \xi_1, \xi_2, \dots\}$ , we say that a class  $\mathcal{K}$  of d.f.s is closed with respect to the random convolution if the condition  $F_\xi \in \mathcal{K}$  implies  $F_{S_\eta} \in \mathcal{K}$ . The first result on the convolution closure of subexponential distributions was obtained by Embrechts and Goldie (see Thm. 4.2 in [11]) and by Cline (see Thm. 2.13 in [6]).

**Theorem 1.** *Let  $\{\xi_1, \xi_2, \dots\}$  be independent copies of a nonnegative r.v.  $\xi$  with subexponential d.f.  $F_\xi$ . Let  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . If  $\mathbb{E}(1 + \delta)^\eta < \infty$  for some  $\delta > 0$ , then the d.f.  $F_{S_\eta} \in \mathcal{S}$ .*

The random closure results for class  $\mathcal{D}$  can be found in [7, 16], and for the class  $\mathcal{L}$ , in [1, 16, 19, 20]. The random closure results for the class  $\mathcal{C}$  can be derived from the results of [14]. We note that in [7, 14, 20], the case of not necessarily identically



**Fig. 1.** Classes of heavy-tailed distributions

distributed r.v.s  $\{\xi_1, \xi_2, \dots\}$  was considered. We further recall two known results. First, in Theorem 2, we give conditions for  $F_{S_\eta}$  to belong to the class  $\mathcal{D}$ . Its proof can be found in [7, Thm. 2.1]. Recall that a d.f.  $F$  belongs to the class  $\mathcal{D}$  if and only if its upper Matuszewska index  $J_F^+ < \infty$ , where by definition

$$J_F^+ = - \lim_{y \rightarrow \infty} \frac{1}{\log y} \log \left( \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \right).$$

**Theorem 2.** *Let r.v.s  $\{\xi_1, \xi_2, \dots\}$  be nonnegative independent but not necessary identically distributed, and  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . Then the d.f.  $F_{S_\eta}$  belongs to the class  $\mathcal{D}$  if the following three conditions are satisfied:*

- (i)  $F_{\xi_\kappa} \in \mathcal{D}$  for some  $\kappa \in \text{supp}(\eta)$ ,
- (ii)  $\limsup_{x \rightarrow \infty} \sup_{n \geq \kappa} \frac{1}{n \overline{F}_{\xi_\kappa}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) < \infty$ ,
- (iii)  $\mathbb{E}\eta^{p+1} < \infty$  for some  $p > J_{F_{\xi_\kappa}}^+$ .

A similar result for r.v.s having d.f.s with consistently varying tails can be found in [14, Thm. 6].

**Theorem 3.** *Let  $\{\xi_1, \xi_2, \dots\}$  be independent real-valued r.v.s, and  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . Then the d.f.  $F_{S_\eta}$  belongs to the class  $\mathcal{C}$  if the following conditions are satisfied:*

- (a)  $F_{\xi_1} \in \mathcal{C}$ ,
- (b) for each  $k \geq 2$ , either  $F_{\xi_k} \in \mathcal{C}$  or  $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$ ,
- (c)  $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \overline{F}_{\xi_1}(x)} \sum_{i=1}^n \overline{F}_{\xi_i}(x) < \infty$ ,
- (d)  $\mathbb{E}\eta^{p+1} < \infty$  for some  $p > J_{F_{\xi_1}}^+$ .

In this work, we consider the randomly stopped maximum  $\xi_{(\eta)}$  and randomly stopped maximum of sums  $S_{(\eta)}$  of independent but not necessarily identically dis-

tributed r.v.s. As was noted before, we restrict our consideration to the class  $\mathcal{C}$  but extend it to the real-valued r.v.s as in Theorem 3.

If r.v.s  $\{\xi_1, \xi_2, \dots\}$  are not identically distributed, then different collections of conditions on r.v.s  $\{\xi_1, \xi_2, \dots\}$  and  $\eta$  imply that  $F_{\xi_{(\eta)}} \in \mathcal{C}$  or  $F_{S_{(\eta)}} \in \mathcal{C}$ . We suppose that some r.v.s from  $\{\xi_1, \xi_2, \dots\}$  have distributions belonging to the class  $\mathcal{C}$ , and we find conditions for r.v.s  $\{\xi_1, \xi_2, \dots\}$  and  $\eta$  such that the distribution of the randomly stopped maximum or the randomly stopped maximum of sums remains in the same class. The results presented and their proofs are closely related to the results of the papers [7, 8, 14].

It is worth noting that the closure properties for d.f.s  $F_{S_{(\eta)}}$  in the case of i.i.d. r.v.s can be derived, for instance, from the asymptotic formulas obtained in [9, 13, 17, 18, 21]. Unfortunately, in the case of nonidentically distributed r.v.s, similar asymptotic formulas do not exist. Therefore we have to use other methods to prove our main results.

The rest of the paper is organized as follows. In Section 2, we present our main results together with a few examples of randomly stopped maximum  $\xi_{(\eta)}$  and randomly stopped maximum of sums  $S_{(\eta)}$  with d.f.s having consistently varying tails. Section 3 is a collection of auxiliary lemmas, and the proofs of the main results are presented in Section 4.

## 2 Main results

In this section, we present four statements, two theorems and two corollaries. Theorem 4 and Corollary 1 deal with the belonging of the d.f.  $F_{\xi_{(\eta)}}$  to the class  $\mathcal{C}$ .

**Theorem 4.** *Let  $\{\xi_1, \xi_2, \dots\}$  be a sequence of independent real-valued r.v.s, and let  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . The d.f. of the randomly stopped maximum  $F_{\xi_{(\eta)}}$  belongs to the class  $\mathcal{C}$  if the following conditions hold:*

- (a)  $F_{\xi_{\varkappa}} \in \mathcal{C}$  for some  $\varkappa \in \text{supp}(\eta)$ ,
- (b) for each  $k \neq \varkappa$ , either  $F_{\xi_k} \in \mathcal{C}$  or  $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_{\varkappa}}(x))$ ,
- (c)  $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{\varphi(n) \overline{F}_{\xi_{\varkappa}}(x)} \sum_{k=1}^n \overline{F}_{\xi_k}(x) < \infty$ ,

where  $\{\varphi(n)\}_{n=1}^{\infty}$  is a positive sequence such that  $\mathbb{E}(\varphi(\eta) \mathbb{1}_{[1, \infty)}(\eta)) < \infty$ .

If r.v.s  $\{\xi_1, \xi_2, \dots\}$  are identically distributed with common d.f.  $F_{\xi} \in \mathcal{C}$ , then conditions (a), (b), and (c) are satisfied with  $\varphi(n) = n$  for  $n \in \mathbb{N}$ . Hence, the following statement immediately follows from Theorem 4.

**Corollary 1.** *Let  $\{\xi_1, \xi_2, \dots\}$  be a sequence of i.i.d. real-valued r.v.s with common d.f.  $F_{\xi} \in \mathcal{C}$ , and let  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . The d.f. of the randomly stopped maximum  $F_{\xi_{(\eta)}}$  belongs to the class  $\mathcal{C}$  if  $\mathbb{E}\eta$  is finite.*

In Theorem 5 and Corollary 2, we present conditions under which the d.f. of the randomly stopped maximum of sums  $S_{(\eta)}$  has consistently varying tail.

**Theorem 5.** *Let  $\{\xi_1, \xi_2, \dots\}$  be independent real-valued r.v.s, and let  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . The d.f.  $F_{S_{(\eta)}}$  belongs to the class  $\mathcal{C}$  if the following conditions hold:*

- (a)  $F_{\xi_k} \in \mathcal{C}$  for each  $k \in \mathbb{N}$ ,
- (b)  $\limsup_{x \rightarrow \infty} \sup_{n \geq 1} \frac{1}{n \overline{F}_{\xi_1}(x)} \sum_{k=1}^n \overline{F}_{\xi_k}(x) < \infty$ ,
- (c)  $\mathbb{E}\eta^{p+1} < \infty$  for some  $p > J_{F_{\xi_1}}^+$ .

Similarly to Corollary 1, we can state the following corollary. We note that in the i.i.d. case condition (b) is obviously satisfied if  $\overline{F}_{\xi_1}(x) > 0$  for all  $x \in \mathbb{R}$ .

**Corollary 2.** *Let  $\{\xi_1, \xi_2, \dots\}$  be i.i.d. real-valued r.v.s with common d.f.  $F_\xi \in \mathcal{C}$ , and let  $\eta$  is a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ . The d.f.  $F_{S(\eta)}$  belongs to the class  $\mathcal{C}$  if  $\mathbb{E}\eta^{p+1} < \infty$  for some  $p > J_{F_\xi}^+$ .*

Further in this section, we present two examples of r.v.s  $\{\xi_1, \xi_2, \dots\}$  and  $\eta$  showing the applicability of our theorems.

**Example 1.** Let  $\{\xi_1, \xi_2, \dots\}$  be independent r.v.s such that:

- r.v.s  $\xi_k$  are exponentially distributed for  $k \equiv 0 \pmod{3}$ , that is,

$$\overline{F}_{\xi_k}(x) = \mathbb{1}_{(-\infty, 0)}(x) + e^{-x} \mathbb{1}_{[0, \infty)}(x), \quad k \in \{3, 6, 9, \dots\};$$

- r.v.s  $\xi_k$  are degenerate at zero for all  $k \equiv 2 \pmod{3}$  and  $k \equiv 1 \pmod{3}, k \geq 4$ , that is,

$$F_{\xi_k}(x) = \mathbb{1}_{[0, \infty)}(x), \quad k \in \{2, 5, 8, \dots\} \cup \{4, 7, 10, \dots\};$$

- $\xi_1 = (1+U)2^{\mathcal{G}}$ , where r.v.s  $U$  and  $\mathcal{G}$  are independent,  $U$  is uniformly distributed on the interval  $[0, 1]$ , and  $\mathcal{G}$  is geometrically distributed with parameter  $q \in (0, 1)$ , that is,

$$\mathbb{P}(\mathcal{G} = l) = (1 - q)q^l, \quad l \in \{0, 1, 2, \dots\}.$$

In addition, let  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$ .

Theorem 4 implies that the d.f. of the randomly stopped maximum  $\xi_{(\eta)}$  belongs to the class  $\mathcal{C}$  if  $1 \in \text{supp}(\eta)$  and  $\mathbb{E}\eta < \infty$  because:

- $F_{\xi_1} \in \mathcal{C}$ , but  $F_{\xi_1} \notin \mathcal{R}$  due to considerations in pp. 122–123 of [2],
- $\overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_1}(x))$  for each  $k \neq 1$ ,
- $\sup_{n \geq 1} \frac{1}{n \overline{F}_{\xi_1}(x)} \sum_{k=1}^n \overline{F}_{\xi_k}(x) \leq 2$  for sufficiently large  $x$ .

**Example 2.** Let  $\{\xi_1, \xi_2, \dots\}$  be independent r.v.s distributed according to Pareto-type laws, namely,

$$\overline{F}_{\xi_k}(x) = \mathbb{1}_{(-\infty, 0)}(x) + \frac{1}{(1+x)^3} \mathbb{1}_{[0, \infty)}(x) \quad \text{if } k \in \{1, 3, 5, \dots\},$$

$$\overline{F}_{\xi_k}(x) = \mathbb{1}_{(-\infty, -1)}(x) + \frac{1}{(2+x)^3} \mathbb{1}_{[-1, \infty)}(x) \quad \text{if } k \in \{2, 4, 6, \dots\}.$$

Further, let  $\eta$  be a counting r.v. independent of  $\{\xi_1, \xi_2, \dots\}$  that has the zeta distribution with parameter 6, that is,

$$\mathbb{P}(\eta = m) = \frac{1}{\zeta(6)} \frac{1}{(m+1)^6}, \quad m \in \{0, 1, 2, \dots\},$$

where  $\zeta$  denotes the Riemann zeta function.

Theorem 5 implies that the d.f. of the randomly stopped maximum of sums  $S_{(\eta)}$  belongs to the class  $\mathcal{C}$  because:

- $F_{\xi_k} \in \mathcal{R} \subset \mathcal{C}$  for each  $k \in \mathbb{N}$ ;
- $\sup_{n \geq 1} \frac{1}{n F_{\xi_1}(x)} \sum_{k=1}^n \bar{F}_{\xi_k}(x) = 1$  for positive  $x$ ;
- $J_{F_{\xi_1}}^+ = 3$  and  $\mathbb{E}\eta^{4.5} < \infty$ .

### 3 Auxiliary lemmas

In this section, we give auxiliary lemmas, which we use in the proof of Theorem 5. The first lemma was proved in [17, Thm. 2.1]; a more general case can be found in [3, Thm. 2.1]. We recall only that  $\mathcal{C} \subset \mathcal{L}$ . Hence, the statement of the next lemma holds for d.f.s with consistently varying tails.

**Lemma 1.** *Let  $\{X_1, X_2, \dots, X_n\}$  be independent real-valued r.v.s such that  $F_{X_k} \in \mathcal{L}$  for all  $k \in \mathbb{N}$ . Then*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i > x\right) \underset{x \rightarrow \infty}{\sim} \mathbb{P}\left(\sum_{i=1}^n X_i > x\right).$$

The second auxiliary lemma was proved in [14, Lemma 3]. It describes the situation where the d.f. of sums of independent r.v.s belongs to the class  $\mathcal{C}$ .

**Lemma 2.** *Let  $\{X_1, X_2, \dots, X_n\}$  be independent real-valued r.v.s. The d.f. of the sum  $\Sigma_n := X_1 + X_2 + \dots + X_n$  belongs to the class  $\mathcal{C}$  if the following two conditions are satisfied:*

- (a)  $F_{X_1} \in \mathcal{C}$ ,
- (b) for each  $k \in \{2, \dots, n\}$ , either  $F_{X_k} \in \mathcal{C}$  or  $\bar{F}_{X_k}(x) = o(\bar{F}_{X_1}(x))$ .

The following statement was proved in [7, Lemma 3.2]. It gives an upper estimate for the tail of the sum of r.v.s from the class  $\mathcal{D}$ .

**Lemma 3.** *Let  $\{X_1, X_2, \dots\}$  be nonnegative independent r.v.s,  $F_{X_v} \in \mathcal{D}$  for some  $v \geq 1$  such that*

$$\limsup_{x \rightarrow \infty} \sup_{n \geq v} \frac{1}{n \bar{F}_{X_v}(x)} \sum_{i=1}^n \bar{F}_{X_i}(x) < \infty.$$

Then, for each  $p > J_{F_{X_v}}^+$ , there exists a positive constant  $c_1$  such that

$$\overline{F}_{\Sigma_n}(x) \leq c_1 n^{p+1} \overline{F}_{X_v}(x),$$

for all  $n \geq v$  and  $x \geq 0$ , where  $F_{\Sigma_n}$  is the d.f. of the sum  $\Sigma_n = X_1 + \dots + X_n$ .

The last useful lemma is an obvious conclusion from Theorem 3.1 in [4].

**Lemma 4.** Let  $\{X_1, X_2, \dots, X_n\}$  be independent real-valued r.v.s. If  $F_{X_k} \in \mathcal{C}$  for each  $k \in \{1, 2, \dots, n\}$ , then

$$\mathbb{P}\left(\sum_{i=1}^n X_i > x\right) \sim \sum_{i=1}^n \overline{F}_{X_i}(x).$$

#### 4 Proofs of the main results

**Proof of Theorem 4.** It suffices to prove that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi(\eta)}(xy)}{\overline{F}_{\xi(\eta)}(x)} \leq 1. \quad (2)$$

For all  $x > 0$  and  $K \in \mathbb{N}$ , we have

$$\begin{aligned} \overline{F}_{\xi(\eta)}(x) &= \sum_{n=1}^{\infty} \overline{F}_{\xi(n)}(x) \mathbb{P}(\eta = n) \\ &= \left( \sum_{n=1}^K + \sum_{n=K+1}^{\infty} \right) \mathbb{P}\left(\bigcup_{k=1}^n \{\xi_k > x\}\right) \mathbb{P}(\eta = n). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\overline{F}_{\xi(\eta)}(xy)}{\overline{F}_{\xi(\eta)}(x)} &= \frac{\sum_{n=1}^K \mathbb{P}\left(\bigcup_{k=1}^n \{\xi_k > xy\}\right) \mathbb{P}(\eta = n)}{\mathbb{P}(\xi(\eta) > x)} \\ &\quad + \frac{\sum_{n=K+1}^{\infty} \mathbb{P}\left(\bigcup_{k=1}^n \{\xi_k > xy\}\right) \mathbb{P}(\eta = n)}{\mathbb{P}(\xi(\eta) > x)} \\ &=: \mathcal{J}_1 + \mathcal{J}_2, \end{aligned} \quad (3)$$

for all  $x > 0$  and  $y \in (0, 1)$ .

Denote

$$\mathcal{K} := \{k \in \mathbb{N} : F_{\xi_k} \notin \mathcal{C} \text{ and } \overline{F}_{\xi_k}(x) = o(\overline{F}_{\xi_{\varkappa}}(x))\}.$$

If  $x > 0$ ,  $1/2 \leq y < 1$ , and  $K \geq \varkappa$ , then

$$\mathcal{J}_1 \leq \frac{1}{\overline{F}_{\xi(\eta)}(x)} \sum_{n=1}^K \sum_{\substack{k=1 \\ k \notin \mathcal{K}}}^n \overline{F}_{\xi_k}(xy) \mathbb{P}(\eta = n) + \sum_{n=1}^K \sum_{\substack{k=1 \\ k \in \mathcal{K}}}^n \frac{\overline{F}_{\xi_k}(x/2)}{\overline{F}_{\xi(\eta)}(x)} \mathbb{P}(\eta = n). \quad (4)$$



If  $k \in \mathcal{K}$ , then

$$\overline{F}_{\xi_k}(x/2) = o(\overline{F}_{\xi_{(\eta)}}(x)), \quad (5)$$

because

$$\frac{\overline{F}_{\xi_k}(x/2)}{\overline{F}_{\xi_{(\eta)}}(x)} \leq \frac{\overline{F}_{\xi_k}(x/2)}{\overline{F}_{\xi_{(\varkappa)}}(x)\mathbb{P}(\eta = \varkappa)} \leq \frac{\overline{F}_{\xi_k}(x/2)}{\overline{F}_{\xi_{\varkappa}}(x/2)P(\eta = \varkappa)} \frac{\overline{F}_{\xi_{\varkappa}}(x/2)}{\overline{F}_{\xi_{\varkappa}}(x)},$$

and  $F_{\xi_k} \in \mathcal{C} \subset \mathcal{D}$ .

The obtained asymptotic relation (5) and estimate (4) imply

$$\begin{aligned} \limsup_{x \rightarrow \infty} \mathcal{J}_1 &\leq \limsup_{x \rightarrow \infty} \frac{1}{\overline{F}_{\xi_{(\eta)}}(x)} \sum_{n=1}^K \sum_{\substack{k=1 \\ k \notin \mathcal{K}}}^n \overline{F}_{\xi_k}(xy) \mathbb{P}(\eta = n) \\ &\leq \limsup_{x \rightarrow \infty} \left( \max_{\substack{1 \leq k \leq K \\ k \notin \mathcal{K}}} \left\{ \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \right\} \frac{1}{\overline{F}_{\xi_{(\eta)}}(x)} \sum_{n=1}^K \sum_{\substack{k=1 \\ k \notin \mathcal{K}}}^n \overline{F}_{\xi_k}(x) \mathbb{P}(\eta = n) \right) \\ &\leq \max_{\substack{1 \leq k \leq K \\ k \notin \mathcal{K}}} \left\{ \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \right\} \\ &\quad \times \limsup_{x \rightarrow \infty} \left\{ \frac{1}{\overline{F}_{\xi_{(\eta)}}(x)} \sum_{n=1}^K \sum_{k=1}^n \overline{F}_{\xi_k}(x) \mathbb{P}(\eta = n) \right\}. \end{aligned} \quad (6)$$

For each  $1 \leq n \leq K$ , we have

$$\overline{F}_{\xi_{(n)}}(x) = \mathbb{P}\left(\bigcup_{k=1}^n \{\xi_k > x\}\right) \geq \sum_{k=1}^n \overline{F}_{\xi_k}(x) \left(1 - \sum_{k=1}^K \overline{F}_{\xi_k}(x)\right),$$

due to the Bonferroni inequality. This implies that, for an arbitrary  $\varepsilon > 0$ ,

$$\sum_{k=1}^n \overline{F}_{\xi_k}(x) \leq (1 + \varepsilon) \overline{F}_{\xi_{(n)}}(x),$$

if  $x$  is sufficiently large.

Substituting the last estimate into inequality (6), we get

$$\begin{aligned} \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{J}_1 &\leq (1 + \varepsilon) \max_{\substack{1 \leq k \leq K \\ k \notin \mathcal{K}}} \left\{ \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi_k}(xy)}{\overline{F}_{\xi_k}(x)} \right\} \\ &= 1 + \varepsilon, \end{aligned}$$

because  $F_{\xi_k} \in \mathcal{C}$  for each  $k \notin \mathcal{K}$ .

Since  $\varepsilon > 0$  is arbitrary, the last estimate implies that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{J}_1 \leq 1. \quad (7)$$

Since

$$\mathbb{P}(\xi_{(\eta)} > x) \geq \mathbb{P}(\xi_{(\varkappa)} > x)\mathbb{P}(\eta = \varkappa) \geq \mathbb{P}(\xi_{\varkappa} > x)\mathbb{P}(\eta = \varkappa),$$

we obtain

$$\mathcal{J}_2 \leq \frac{\sum_{n=K+1}^{\infty} \sum_{k=1}^n \overline{F}_{\xi_k}(xy)\mathbb{P}(\eta = n)}{\overline{F}_{\xi_{\varkappa}}(x)\mathbb{P}(\eta = \varkappa)},$$

for all  $x > 0$  and  $y \in (0, 1)$ .

Condition (c) of the theorem implies that, for some constant  $c_2 > 0$ ,

$$\sum_{k=1}^n \overline{F}_{\xi_k}(x) \leq c_2 \varphi(n) \overline{F}_{\xi_{\varkappa}}(x),$$

for all sufficiently large  $x$  and all  $n \geq 1$ .

Consequently,

$$\begin{aligned} & \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{J}_2 \\ & \leq \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{c_2 \overline{F}_{\xi_{\varkappa}}(xy) \sum_{n=K+1}^{\infty} \varphi(n)\mathbb{P}(\eta = n)}{\overline{F}_{\xi_{\varkappa}}(x)\mathbb{P}(\eta = \varkappa)} \\ & = \frac{c_2}{\mathbb{P}(\eta = \varkappa)} \left( \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi_{\varkappa}}(xy)}{\overline{F}_{\xi_{\varkappa}}(x)} \right) \sum_{n=K+1}^{\infty} \varphi(n)\mathbb{P}(\eta = n). \end{aligned} \quad (8)$$

Relations (3), (7), and (8) imply that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi_{(\eta)}}(xy)}{\overline{F}_{\xi_{(\eta)}}(x)} \leq 1 + \frac{c_2}{\mathbb{P}(\eta = \varkappa)} \mathbb{E}(\varphi(\eta)\mathbb{1}_{[K+1, \infty)}(\eta)),$$

for arbitrary  $K \geq \varkappa$ .

The desired inequality (2) now follows from the last estimate because  $\mathbb{E}(\varphi(\eta)\mathbb{1}_{[1, \infty)}(\eta))$  is finite due to the conditions of the theorem. Theorem 4 is proved.  $\square$

**Proof of Theorem 5.** Similarly as in the proof of Theorem 4, it suffices to show that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_{(\eta)}}(xy)}{\overline{F}_{S_{(\eta)}}(x)} \leq 1. \quad (9)$$

If  $K \in \mathbb{N}$  and  $x > 0$ , then

$$\mathbb{P}(S_{(\eta)} > x) = \left( \sum_{n=1}^K + \sum_{n=K+1}^{\infty} \right) \mathbb{P}(S_{(n)} > x)\mathbb{P}(\eta = n).$$

Therefore,

$$\frac{\mathbb{P}(S_{(\eta)} > xy)}{\mathbb{P}(S_{(\eta)} > x)} = \frac{\sum_{n=1}^K \mathbb{P}(S_{(n)} > xy)\mathbb{P}(\eta = n)}{\mathbb{P}(S_{(\eta)} > x)}$$

$$\begin{aligned}
 & + \frac{\sum_{n=K+1}^{\infty} \mathbb{P}(S_{(n)} > xy) \mathbb{P}(\eta = n)}{\mathbb{P}(S_{(\eta)} > x)} \\
 =: & \mathcal{I}_1 + \mathcal{I}_2,
 \end{aligned} \tag{10}$$

for all  $x > 0$  and  $y \in (0, 1)$ .

The r.v.  $\eta$  is not degenerate at zero. Therefore, there exists  $a \in \mathbb{N}$  such that  $\mathbb{P}(\eta = a) > 0$ . If  $K \geq a$ , then

$$\mathcal{I}_1 \leq \frac{\sum_{n=1}^K \mathbb{P}(S_{(n)} > xy) \mathbb{P}(\eta = n)}{\sum_{n=1}^K \mathbb{P}(S_{(n)} > x) \mathbb{P}(\eta = n)} \leq \max_{\substack{1 \leq n \leq K \\ n \in \text{supp}(\eta)}} \frac{\mathbb{P}(S_{(n)} > xy)}{\mathbb{P}(S_{(n)} > x)},$$

due to the inequality

$$\frac{a_1 + a_2 + \dots + a_m}{b_1 + b_2 + \dots + b_m} \leq \max \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_m}{b_m} \right\},$$

provided for all  $a_i \geq 0$ ,  $b_i > 0$ ,  $i \in \{1, 2, \dots, m\}$ , and  $m \in \mathbb{N}$ .

Since  $\mathcal{C} \subset \mathcal{L}$ , using Lemma 1, we obtain

$$\begin{aligned}
 & \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{I}_1 \\
 & \leq \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \max_{\substack{1 \leq n \leq K \\ n \in \text{supp}(\eta)}} \frac{\overline{F}_{S_{(n)}}(xy)}{\overline{F}_{S_{(n)}}(x)} \\
 & = \max_{\substack{1 \leq n \leq K \\ n \in \text{supp}(\eta)}} \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_{(n)}}(xy)}{\overline{F}_{S_{(n)}}(xy)} \frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)} \frac{\overline{F}_{S_n}(x)}{\overline{F}_{S_{(n)}}(x)} \\
 & \leq \max_{1 \leq n \leq K} \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{S_n}(xy)}{\overline{F}_{S_n}(x)}.
 \end{aligned}$$

According to Lemma 2, the d.f.  $F_{S_n}$  belongs to the class  $\mathcal{C}$  for each fixed  $n$ . Therefore,

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{I}_1 \leq 1. \tag{11}$$

Let us now consider the term  $\mathcal{I}_2$  from expression (10). If  $x > 0$ , then the numerator of  $\mathcal{I}_2$  can be estimated as follows:

$$\begin{aligned}
 & \sum_{n=K+1}^{\infty} \mathbb{P}(S_{(n)} > xy) \mathbb{P}(\eta = n) \\
 & = \sum_{n=K+1}^{\infty} \mathbb{P}(\max\{S_1, \dots, S_n\} > xy) \mathbb{P}(\eta = n) \\
 & \leq \sum_{n=K+1}^{\infty} \mathbb{P}(\max\{S_1^{(+)}, \dots, S_n^{(+)}\} > xy) \mathbb{P}(\eta = n)
 \end{aligned}$$

$$= \sum_{n=K+1}^{\infty} \mathbb{P}(S_n^{(+)} > xy) \mathbb{P}(\eta = n),$$

where  $S_k^{(+)} := \xi_1^+ + \dots + \xi_k^+$  and  $\xi^+ := \xi \mathbb{1}_{(0, \infty)}(\xi)$ .

We can apply Lemma 3 for the last sum because of condition (b) of the theorem and the fact that  $F_{\xi_1} \in \mathcal{C} \Rightarrow F_{\xi_1^+} \in \mathcal{D}$ . Using this lemma, we get

$$\sum_{n=K+1}^{\infty} \mathbb{P}(S_n > xy) \mathbb{P}(\eta = n) \leq c_3 \overline{F}_{\xi_1}(xy) \sum_{n=K+1}^{\infty} n^{p+1} \mathbb{P}(\eta = n), \quad (12)$$

for some positive constant  $c_3$  and for all  $x > 0$  and  $y \in (0, 1)$ .

On the other hand, for the denominator of  $\mathcal{I}_2$ , we have

$$\begin{aligned} \mathbb{P}(S_{(\eta)} > x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x) \mathbb{P}(\eta = n) \\ &\geq \mathbb{P}(\max\{S_1, \dots, S_a\} > x) \mathbb{P}(\eta = a) \\ &\geq \mathbb{P}(S_a > x) \mathbb{P}(\eta = a). \end{aligned}$$

According to the conditions of the theorem, the d.f.  $F_{\xi_k}$  belongs to the class  $\mathcal{C}$  for each fixed index  $k \in \mathbb{N}$ . Hence, using Lemma 4, we get

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{\mathbb{P}(S_a > x)}{\overline{F}_{\xi_1}(x)} &\geq \liminf_{x \rightarrow \infty} \frac{\mathbb{P}(S_a > x)}{\sum_{i=1}^a \overline{F}_{\xi_i}(x)} \liminf_{x \rightarrow \infty} \frac{\sum_{i=1}^a \overline{F}_{\xi_i}(x)}{\overline{F}_{\xi_1}(x)} \\ &= \liminf_{x \rightarrow \infty} \frac{\sum_{i=1}^a \overline{F}_{\xi_i}(x)}{\overline{F}_{\xi_1}(x)} \geq \liminf_{x \rightarrow \infty} \frac{\overline{F}_{\xi_1}(x)}{\overline{F}_{\xi_1}(x)} = 1. \end{aligned}$$

The last two estimates imply that

$$\mathbb{P}(S_{(\eta)} > x) \geq \frac{1}{2} \overline{F}_{\xi_1}(x) \mathbb{P}(\eta = a), \quad (13)$$

for sufficiently large  $x$ .

Therefore, by inequalities (12) and (13) we have

$$\begin{aligned} &\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \mathcal{I}_2 \\ &\leq \frac{2c_3}{\mathbb{P}(\eta = a)} \left( \limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}_{\xi_1}(xy)}{\overline{F}_{\xi_1}(x)} \right) \sum_{n=K+1}^{\infty} n^{p+1} \mathbb{P}(\eta = n). \quad (14) \end{aligned}$$

Finally, substituting estimates (11) and (14) into (10), we obtain that

$$\limsup_{y \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_{(\eta)} > xy)}{\mathbb{P}(S_{(\eta)} > x)} \leq 1 + \frac{2c_3}{\mathbb{P}(\eta = a)} \mathbb{E}(\eta^{p+1} \mathbb{1}_{[K+1, \infty)}(\eta)),$$

for an arbitrary  $K \geq a$ .

This inequality implies the desired relation (9) due to condition (c) of the theorem. Theorem 5 is proved.  $\square$

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