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# Bonus–malus systems with different claim types and varying deductibles

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**Abstract** The paper deals with bonus–malus systems with different claim types and varying deductibles. The premium relativities are softened for the policyholders who are in the malus zone and these policyholders are subject to per claim deductibles depending on their levels in the bonus–malus scale and the types of the reported claims. We introduce such bonus–malus systems and study their basic properties. In particular, we investigate when it is possible to introduce varying deductibles, what restrictions we have and how we can do this. Moreover, we deal with the special case where varying deductibles are applied to the claims reported by policyholders occupying the highest level in the bonus–malus scale and consider two allocation principles for the deductibles. Finally, numerical illustrations are presented.

**Keywords** Bonus–malus system, claim type, varying deductible, indifference principle, allocation principle, premium relativity, Markov chain, transition matrix, stationary distribution

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## 1 Introduction and motivation

One of the main tasks of an actuary is to design a tariff structure that fairly distributes the total risk of potential losses among policyholders. To this end, he often has to grade all policyholders into risk classes such that all policyholders belonging to the same class pay the same premium. Rating systems penalizing policyholders responsible for one or more accidents by premium surcharges (or maluses), and rewarding claim-free policyholders by giving them discounts (or bonuses) are now in force in many developed countries. Such systems, which are often called bonus–malus systems, aim to assess individual risks better.

The amount of premium is adjusted each year on the basis of the individual claims experience. In practice, a bonus–malus scale consists of a finite number of levels and each of the levels has its own relative premium. After each year, the policyholder moves up or down according to the transition rules and the number of claims reported during the current year. Thus, bonus–malus systems also encourage policyholders to be careful. Note that the premium relativities are traditionally computed using a quadratic loss function. This method is proposed by Norberg in his pioneering work [14] on segmented tariffs. Alternatively, Denuit and Dhaene [2] use an exponential loss function to compute the relativities.

In most of the commercial bonus–malus systems used by insurance companies, knowing the current level and the number of claims during the current period suffices to determine the next level in the scale. Therefore, the future level depends only on the present and not on the past. The numbers of claims in different years are usually assumed to be independent. So the trajectory of each policyholder in the bonus–malus scale can be considered as a Markov chain. For details and more information concerning bonus–malus systems, we refer the reader to [3, 10, 19]. In particular, [3] presents a comprehensive treatment of the various experience rating systems and their relationships with risk classification.

As pointed out by many authors, traditional bonus–malus systems suffer from two considerable drawbacks:

(i) The claim amounts are not taken into account. So a posteriori corrections depend only on the number of claims. In this case, policyholders who had accidents with small or large claims are penalized unfairly in the same way. In particular, this breeds bonus hunger when policyholders cover small claims themselves in order to avoid future premium increases.

(ii) At any time the policyholders may leave the insurance company without any further financial penalties. Thus, bonus–malus systems create the possibility of malus evasion, i.e. the situation when the policyholders leave the insurance company to avoid premium increase because of reported claims.

An alternative approach, which, at least theoretically, eliminates the second drawback, is proposed by Holtan [9]. He suggests the use of very high deductibles that may be borrowed by the policyholders in the insurance company. The deductibles are assumed to be constant for all policyholders, i.e. independent of the level they occupy in the bonus–malus system at the time of claim occurrence. Although technically acceptable, this approach obviously causes considerable practical problems. Practical consequences of Holtan's proposal are investigated by Lemaire and Zi [11]. Particularly, it is shown that the introduction of high deductibles increases the variability of payments and the efficiency of the rating systems for most policyholders.

Bonus–malus systems with varying deductibles are introduced by Pitrebois, Denuit and Walhin [17]. Specifically, the a posteriori premium correction induced by the bonus–malus system is replaced by a deductible (in whole or in part). To each level of the bonus–malus system in the malus zone is assigned an amount of deductible, which is applied to the claims filed during the coverage period either annually or claim by claim. Relative premiums at high levels of bonus–malus systems are often very large and the systems can be softened by introducing deductibles. The insurance company compensates the reduced penalties in the malus zone with the deductibles paid by policyholders who report claims being in the malus zone. This can be commercially attractive since the policyholders are penalized only if they report claims in the future.

As pointed out in [17], combining bonus–malus systems with varying deductibles presents a number of advantages. Firstly, the policyholders will do all their best to prevent or at least decrease the losses. Secondly, even if the policyholder leaves the company after a claim, he has to pay for the deductible. Thirdly, relative premiums and amounts of deductibles may be tuned in an optimal way in order to attract the policyholders. The numerical illustrations show that the mixed case (reduced relative premiums combined with per claim deductibles) gives the best results. The amounts of deductibles are moderate in this case. Nevertheless, this approach does not eliminate the first drawback mentioned above.

To eliminate the first drawback, a few other approaches have been proposed. Bonus–malus systems involving different claim types are designed in [18]. Each claim type induces a specific penalty for the policyholder. In particular, claim amounts can be taken into account in this way. For some early results in this direction, see also [10, 15, 16].

The next approach is based on a generalization of results obtained by Dionne and Vanasse [4, 5], who propose a bonus–malus system that integrates a priori and a posteriori information on an individual basis. Specifically, the system is derived as a function of the years that the policyholder is in the portfolio, the number of claims and his individual characteristics. Frangos and Vrontos [6] expand the frame developed in [4, 5] and propose a generalized bonus–malus system that takes into consideration simultaneously the policyholder's characteristics, the number of his claims and the exact amount of each claim. Particularly, it is assumed that we have all information about the claim frequency history and the claim amount history for each policyholder for the time period he is in the portfolio. In this generalized bonus–malus system, the premium is a function of the years that the policyholder is in the portfolio, his number of claims, the amount of each claim and the significant a priori rating variables. Therefore, the future premium depends on the past (all policyholder's history) and is actually calculated individually for each policyholder. This approach is extended and developed in [12, 13, 20].

Another approach that takes into account claim amounts is proposed by Bonsdorff [1]. The author considers a general framework for a bonus–malus system based on the number of claims during the previous year and the total amount of claims during the previous year. The set of the bonus levels is some interval and the transitions between the levels are determined by these characteristics.

Gómez-Déniz, Hernández-Bastida and Fernández-Sánchez [8] obtain expressions that can be used to compute bonus–malus premiums based on the distribution of the total claim amount but not on the claims which produce the amounts.

Another modification of traditional bonus–malus systems, which take into account only the number of claims, is considered by Gómez-Déniz in the recent paper

[7]. The author presents a statistical model, which distinguishes between two different claim types, incorporating a bivariate distribution based on the assumption of dependence.

The present paper deals with the case where both penalty types are used for the policyholders who are in the malus zone. Specifically, the premium relativities, which are computed using a quadratic loss function, are softened and the policyholders who are in the malus zone are subject to per claim deductibles. The mixed bonus–malus system combining premium relativities and deductibles is expected to be the most relevant in practice (see [3, 17]).

We try to eliminate both drawbacks mentioned above. To take into account claim amounts, we consider different claim types and use the multi-event bonus–malus systems introduced in [18]. To eliminate the second drawback, we introduce varying deductibles for the policyholders who are in the malus zone. The deductibles depend on the level of the policyholder in the bonus–malus scale and the types of the reported claims. Such bonus–malus systems present a number of advantages and seem to be very attractive for policyholders. Namely, policyholders reporting small and large claims are not penalized in the same way. This helps to avoid or at least decrease bonus hunger. Moreover, all advantages mentioned in [17] are also in force.

The rest of the paper is organized as follows. In Section 2, we describe the multi-event bonus–malus systems introduced in [18]. In Section 3, we introduce varying deductibles in such bonus–malus systems and study basic properties of such systems. In particular, we investigate when it is possible to introduce varying deductibles in the bonus–malus systems with different claim types, what restrictions we have and how we can do this. Section 4 deals with the special case where varying deductibles are applied to the claims reported by policyholders occupying the highest level in the bonus–malus scale. We consider two allocation principles for the deductibles, which seems to be natural and fair for policyholders. In Section 5, we consider an example of such a bonus–malus system, deal with exponentially distributed claim sizes and present numerical illustrations. Section 6 completes the paper.

## 2 Bonus–malus systems with different claim types

To take into consideration different claim types, we use the multi-event bonus–malus systems introduced in [18] (see also [3]).

Let us pick a policyholder at random from the portfolio and denote by $N$ the number of claims reported by the policyholder during the year. In what follows, we assume that there is no a priori risk classification (or we work inside a specified rating cell). Denote by $\lambda > 0$ the a priori annual expected claim frequency. Let $\Theta$ be the (unknown) accident proneness of this policyholder, i.e. $\Theta$ represents the residual effect of unobserved characteristics. The risk profile of the portfolio is described by the distribution function $F_\Theta$ of $\Theta$. It is usually assumed that $\mathbb{P}[\Theta \geq 0] = 1$ and $\mathbb{E}[\Theta] = 1$. Thus, the actual (unknown) annual claim frequency of this policyholder is $\lambda\Theta$.

We assume that the number of claims $N$ is mixed-Poisson distributed. To be more precise, the conditional probability mass function of $N$ is given by

$$\mathbb{P}[N = j \mid \Theta = \theta] = \frac{(\lambda\theta)^j}{j!} e^{-\lambda\theta}, \quad j \geq 0.$$

Hence, the unconditional probability mass function of $N$ is given by

$$\mathbb{P}[N = j] = \int_0^{+\infty} \mathbb{P}[N = j \mid \Theta = \theta] \,\mathrm{d}F_\Theta(\theta), \quad j \geq 0.$$

We introduce $m+1$ different claim types reported by the policyholder. Each claim type induces a specific penalty for the policyholder, which will be described below. The type of the given claim is determined by the claim amount $C$. We assume that all claim amounts are independent and identically distributed, and claim amounts and claim frequencies are mutually independent. The claims are classified according to a multinomial scheme. Let $0 < c_1^* < c_2^* < \cdots < c_m^* < \infty$. We choose these numbers so that all claims of size less than or equal to $c_1^*$ are considered as claims of type 0, all claims of size from the interval $(c_1^*, c_2^*]$ are considered as claims of type 1 and so on; finally, all claims of size greater than $c_m^*$ are considered as claims of type $m$. Let

$$q_0 = \mathbb{P}\big[C \leq c_1^*\big], \qquad q_1 = \mathbb{P}\big[c_1^* < C \leq c_2^*\big],$$
$$q_2 = \mathbb{P}\big[c_2^* < C \leq c_3^*\big], \quad \ldots, \quad q_m = \mathbb{P}\big[C > c_m^*\big].$$

Thus, each time a claim is reported, it is classified in one of the $m+1$ possible categories with probabilities $q_0, q_1, \ldots, q_m$. It is clear that $\sum_{i=0}^m q_i = 1$. Moreover, it is natural to choose the $c_i^*$ so that all $q_i$ are strictly positive.

Denote by $N_i$ the number of claims of type $i$, $0 \leq i \leq m$. Therefore, for a given $\Theta$, the random variables $N_0, N_1, \ldots, N_m$ are mutually independent and the corresponding conditional probability mass function is given by

$$\mathbb{P}[N_i = j \mid \Theta = \theta] = \frac{(\lambda\theta q_i)^j}{j!} e^{-\lambda\theta q_i}, \quad j \geq 0,\ 0 \leq i \leq m. \tag{1}$$

The bonus–malus scale is assumed to have $s+1$ levels numbered from 0 to $s$. A higher level number indicates a higher premium. In particular, the policyholders who are at level 0 enjoy the maximal bonus. A specified level is assigned to a new policyholder. Each claim-free year is rewarded by a bonus point, i.e. the policyholder goes one level down. Each claim type entails a specific penalty expressed as a fixed number of levels per claim. It is natural to assume that larger claims entail more severe penalties.

We suppose that knowing the present level and the number of claims of each type filed during the present year suffices to determine the level to which the policyholder is transferred. So the bonus–malus system may be represented by a Markov chain.

Let $p_{l_0 l}(\lambda\theta; \mathbf{q})$ be the probability of moving from level $l_0$ to level $l$ for a policyholder with annual mean claim frequency $\lambda\theta$ and vector of probabilities $\mathbf{q} = (q_0, q_1, \ldots, q_m)^T$, where $0 \leq l_0 \leq s$, $0 \leq l \leq s$ and $q_i$ is the probability that the claim is of type $i$. Denote by $P(\lambda\theta; \mathbf{q})$ the one-step transition matrix, i.e.

$$P(\lambda\theta; \mathbf{q}) = \begin{pmatrix} p_{00}(\lambda\theta; \mathbf{q}) & p_{01}(\lambda\theta; \mathbf{q}) & \ldots & p_{0s}(\lambda\theta; \mathbf{q}) \\ p_{10}(\lambda\theta; \mathbf{q}) & p_{11}(\lambda\theta; \mathbf{q}) & \ldots & p_{1s}(\lambda\theta; \mathbf{q}) \\ \vdots & \vdots & \ddots & \vdots \\ p_{s0}(\lambda\theta; \mathbf{q}) & p_{s1}(\lambda\theta; \mathbf{q}) & \ldots & p_{ss}(\lambda\theta; \mathbf{q}) \end{pmatrix}.$$

Taking the $n$th power of $P(\lambda\theta; \mathbf{q})$ yields the $n$-step transition matrix with elements $p_{l_0 l}^{(n)}(\lambda\theta; \mathbf{q})$. Here $p_{l_0 l}^{(n)}(\lambda\theta; \mathbf{q})$ is the probability of moving from level $l_0$ to level $l$ in $n$ transitions.

The transition matrix $P(\lambda\theta; \mathbf{q})$ is assumed to be regular, i.e. there is some integer $n_0 \geq 1$ such that all entries of $P((\lambda\theta; \mathbf{q}))^{n_0}$ are strictly positive. Consequently, the Markov chain describing the trajectory of a policyholder with expected claim frequency $\lambda\theta$ and vector of probabilities $\mathbf{q}$ is ergodic and thus possesses a stationary distribution

$$\boldsymbol{\pi}(\lambda\theta; \mathbf{q}) = \big(\pi_0(\lambda\theta; \mathbf{q}), \pi_1(\lambda\theta; \mathbf{q}), \ldots, \pi_s(\lambda\theta; \mathbf{q})\big)^T.$$

Here $\pi_l(\lambda\theta; \mathbf{q})$ is the stationary probability for a policyholder with annual mean claim frequency $\lambda\theta$ to be at level $l$, i.e.

$$\pi_l(\lambda\theta; \mathbf{q}) = \lim_{n\to\infty} p_{l_0 l}^{(n)}(\lambda\theta; \mathbf{q}).$$

The stationary probabilities $\pi_l(\lambda\theta; \mathbf{q})$ can be obtained directly (see [3, 19]). Indeed, since the matrix $P(\lambda\theta; \mathbf{q})$ is regular, the matrix $I - P(\lambda\theta; \mathbf{q}) + E$ is invertible and $\boldsymbol{\pi}(\lambda\theta; \mathbf{q})$ is given by

$$\boldsymbol{\pi}^T(\lambda\theta; \mathbf{q}) = \mathbf{e}^T \big(I - P(\lambda\theta; \mathbf{q}) + E\big)^{-1}, \tag{2}$$

where $\mathbf{e}$ is a column vector of 1s, $E$ is an $(s+1) \times (s+1)$ matrix consisting of $s+1$ column vectors $\mathbf{e}$ and $I$ is an $(s+1) \times (s+1)$ identity matrix.

Next, let $L$ be the level occupied in the scale by a randomly selected policyholder and $\pi_l$ be the proportion of policyholders at level $l$ once the steady state has been reached. Then

$$\pi_l = \mathbb{P}[L = l] = \int_0^{+\infty} \pi_l(\lambda\theta; \mathbf{q})\, \mathrm{d}F_\Theta(\theta), \quad 0 \leq l \leq s. \tag{3}$$

It is easily seen that $\sum_{l=0}^{s} \pi_l = 1$.

We denote by $r_l$ the premium relativities associated with level $l$. The meaning is that a policyholder occupying level $l$ pays the premium equal to $\lambda r_l \mathbb{E}[C]$. Note that here and below we consider only net premiums. In what follows, we assume that $\mathbb{E}[C] < \infty$ and the distribution function $F_C$ of $C$ is continuous.

To compute the premium relativities $r_l$, we use a quadratic loss function as proposed in [14] (see also [3, 10]). To this end, we minimize the expected squared difference between the "true" relative premium $\Theta$ and the relative premium $r_L$ applicable to this policyholder after the stationary state has been reached, i.e. we minimize $\mathbb{E}[(\Theta - r_L)^2]$. The solution to this problem is given by

$$r_l = \frac{\int_0^{+\infty} \theta \pi_l(\lambda\theta; \mathbf{q})\, \mathrm{d}F_\Theta(\theta)}{\int_0^{+\infty} \pi_l(\lambda\theta; \mathbf{q})\, \mathrm{d}F_\Theta(\theta)} \tag{4}$$

(see, e.g., [3, pp. 185–186] for the details).

## 3 Varying deductibles in the bonus–malus systems with different claim types

The policyholder occupying level $l$ in the bonus–malus systems described in Section 2 should pay $\lambda r_l \mathbb{E}[C]$. We suppose that the relative premiums of the policyholders who are in the bonus zone, i.e. with $r_l \le 1$, are unchanged. So they pay premiums equal to $\lambda r_l \mathbb{E}[C]$ and are subject to no further penalties. Let $s_0 = \min\{l : r_l > 1\}$. The relative premiums of the policyholders who are in the malus zone, i.e. with $r_l > 1$, or equivalently occupying level $l$ such that $s_0 \le l \le s$, are softened in the following way. Instead of paying the premium equal to $\lambda r_l \mathbb{E}[C]$, the policyholder who is at level $l$ pays a reduced premium equal to $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ for some specified $\alpha_l \ge 0$ depending on level $l$. We suppose that $\alpha_l$ satisfy the following assumption.

**Assumption 1.** $1 \le (1-\alpha_{s_0}) r_{s_0} \le (1-\alpha_{s_0+1}) r_{s_0+1} \le \cdots \le (1-\alpha_s) r_s$.

Note that Assumption 1 implies that the policyholder occupying a higher level in the bonus–malus scale pays a higher reduced premium, which is not less than the basic premium $\lambda \mathbb{E}[C]$. Moreover, from Assumption 1 we have $0 \le \alpha_l \le 1 - 1/r_l$ for all $l$ such that $s_0 \le l \le s$. The case $\alpha_l = 0$ corresponds to the situation when the policyholder occupying level $l$ pays premium equal to $\lambda r_l \mathbb{E}[C]$ and is subject to no further penalties. If $\alpha_l = 1 - 1/r_l$, then the policyholder pays only the basic premium $\lambda \mathbb{E}[C]$ and has to pay something for claims in the future. To compensate the reduced premium, the policyholder is subject to per claim deductible, i.e. applied to each reported claim separately, equal to $d_{l,i}$ depending on level $l$ occupied in the malus zone and claim type $i$. We impose the following natural restrictions on $d_{l,i}$.

**Assumption 2.** (i) For all $l$ such that $s_0 \le l \le s$, we have

$$0 \le d_{l,0} \le c_1^*, \qquad 0 \le d_{l,1} \le c_1^*, \qquad 0 \le d_{l,2} \le c_2^*, \qquad \ldots, \qquad 0 \le d_{l,m} \le c_m^*.$$

(ii) For every fixed $l$ such that $s_0 \le l \le s$, we have

$$d_{l,0} \le d_{l,1} \le \cdots \le d_{l,m}.$$

(iii) For every fixed $i$ such that $0 \le i \le m$, we have

$$d_{s_0,i} \le d_{s_0+1,i} \le \cdots \le d_{s,i}.$$

Assertion (i) of Assumption 2 means that if a claim reported is of type $i$, where $0 \le i \le m$, then the deductible applied to it is strictly less than the claim amount, i.e. the insurance company covers at least some part of the losses. Next, assertion (ii) implies that larger claims are subject to higher deductibles. Finally, assertion (iii) means that the higher level in the bonus–malus scale implies a higher deductible for each specified claim amount.

For policyholders occupying level $l$, the deductibles $d_{l,0}, d_{l,1}, \ldots, d_{l,m}$ are found using the indifference principle (see [3, 17]): for this group of policyholders, the part $\alpha_l$ of the penalties induced by the bonus–malus system is on average equal to the total amount of deductibles paid by these policyholders. Thus, the indifference principle for the policyholder occupying level $l$ can be written in the following way:

$$\begin{aligned}\lambda r_l \mathbb{E}[C] = (1-\alpha_l)\lambda r_l \mathbb{E}[C] + \lambda r_l \big(&\mathbb{E}[C \mid C \le d_{l,0}]\, \mathbb{P}[C \le d_{l,0}] \\ &+ d_{l,0}\, \mathbb{P}\big[d_{l,0} < C \le c_1^*\big] + d_{l,1}\, \mathbb{P}\big[c_1^* < C \le c_2^*\big] + \ldots \\ &+ d_{l,m}\, \mathbb{P}\big[C > c_m^*\big]\big), \quad s_0 \le l \le s. \end{aligned} \tag{5}$$

On the left-hand side of (5), we have the premium paid by the policyholder occupying level $l$ in the bonus–malus system described in Section 2. On the right-hand side of (5), we have the expected amount paid by this policyholder in the bonus–malus system with varying deductibles. This amount consists of the reduced premium and the expected amount of penalties induced by deductibles.

Equation (5) can be rewritten as

$$\alpha_l \mathbb{E}[C] = \mathbb{E}[C \mid C \le d_{l,0}]\, \mathbb{P}[C \le d_{l,0}] + d_{l,0}\, \mathbb{P}\big[d_{l,0} < C \le c_1^*\big] \\ + d_{l,1}\, \mathbb{P}\big[c_1^* < C \le c_2^*\big] + \cdots + d_{l,m}\, \mathbb{P}\big[C > c_m^*\big], \quad s_0 \le l \le s,$$

which is equivalent to

$$\alpha_l \mathbb{E}[C] = \mathbb{E}[C \mid C \le d_{l,0}]\, \mathbb{P}[C \le d_{l,0}] + d_{l,0}\, \big(q_0 - \mathbb{P}[C \le d_{l,0}]\big) \\ + d_{l,1} q_1 + \cdots + d_{l,m} q_m, \quad s_0 \le l \le s. \tag{6}$$

Thus, in order to introduce varying deductibles in the bonus–malus system, we have to choose $\alpha_l$, $d_{l,0}$, $d_{l,1}$, ..., $d_{l,m}$ satisfying (6) and Assumptions 1 and 2 for all $l$ such that $s_0 \le l \le s$. In what follows, we call any such combination of $\alpha_l$, $d_{l,0}$, $d_{l,1}$, ..., $d_{l,m}$, where $s_0 \le l \le s$, by a solution to (6).

**Lemma 1.** *The right-hand side of* (6) *is a non-decreasing function of each of the variables* $d_{l,0}$, $d_{l,1}$, ..., $d_{l,m}$.

**Proof.** The assertion of Lemma 1 is evident for variables $d_{l,1}$, ..., $d_{l,m}$. We now show it for $d_{l,0}$. Let $0 \le d'_{l,0} \le d''_{l,0} \le c_1^*$. Then we get

$$\begin{aligned} &\mathbb{E}\big[C \mid C \le d''_{l,0}\big]\, \mathbb{P}\big[C \le d''_{l,0}\big] + d''_{l,0}\, \mathbb{P}\big[d''_{l,0} < C \le c_1^*\big] \\ &\quad - \mathbb{E}\big[C \mid C \le d'_{l,0}\big]\, \mathbb{P}\big[C \le d'_{l,0}\big] - d'_{l,0}\, \mathbb{P}\big[d'_{l,0} < C \le c_1^*\big] \\ &\ge \frac{\mathbb{E}[C \mid C \le d'_{l,0}]\, \mathbb{P}[C \le d'_{l,0}] + d'_{l,0}\, \mathbb{P}[d'_{l,0} < C \le d''_{l,0}]}{\mathbb{P}[C \le d''_{l,0}]}\, \mathbb{P}\big[C \le d''_{l,0}\big] \\ &\quad + d''_{l,0}\, \mathbb{P}\big[d''_{l,0} < C \le c_1^*\big] - \mathbb{E}\big[C \mid C \le d'_{l,0}\big]\, \mathbb{P}\big[C \le d'_{l,0}\big] - d'_{l,0}\, \mathbb{P}\big[d'_{l,0} < C \le c_1^*\big] \\ &= d'_{l,0}\, \mathbb{P}\big[d'_{l,0} < C \le d''_{l,0}\big] + d''_{l,0}\, \mathbb{P}\big[d''_{l,0} < C \le c_1^*\big] - d'_{l,0}\, \mathbb{P}\big[d'_{l,0} < C \le c_1^*\big] \\ &= \big(d''_{l,0} - d'_{l,0}\big)\, \mathbb{P}\big[d''_{l,0} < C \le c_1^*\big] \ge 0, \end{aligned}$$

which proves the lemma. □

In addition, taking into account the continuity of $F_C$ gives that the right-hand side of (6) is continuous in $d_{l,0}$, $d_{l,1}$, ..., $d_{l,m}$.

**Proposition 1.** *For any fixed* $\alpha_l$ *satisfying Assumption 1, we have*

$$d_{l,m} \le \min\big\{\alpha_l \mathbb{E}[C]/q_m,\ c_m^*\big\}, \quad s_0 \le l \le s. \tag{7}$$

**Proof.** From Lemma 1, we conclude that for any fixed $\alpha_l$, the maximum possible value of $d_{l,m}$ is when $d_{l,0} = d_{l,1} = \cdots = d_{l,m-1} = 0$. Therefore, $d_{l,m} \le \alpha_l \mathbb{E}[C]/q_m$. Taking into account assertion (i) of Assumption 2 yields (7). □

**Proposition 2.** *If* $l_1$ *and* $l_2$ *are such that* $s_0 \le l_1 < l_2 \le s$, *then* $\alpha_{l_1} \le \alpha_{l_2}$.

**Proof.** By assertion (iii) of Assumption 2, we have $d_{l_1,0} \le d_{l_2,0}$, $d_{l_1,1} \le d_{l_2,1}$, ..., $d_{l_1,m} \le d_{l_2,m}$. Consequently, from Lemma 1, it follows that

$$\mathbb{E}[C \mid C \le d_{l_1,0}]\,\mathbb{P}[C \le d_{l_1,0}] + d_{l_1,0}\big(q_0 - \mathbb{P}[C \le d_{l_1,0}]\big) + d_{l_1,1}q_1 + \cdots + d_{l_1,m}q_m \le \mathbb{E}[C \mid C \le d_{l_2,0}]\,\mathbb{P}[C \le d_{l_2,0}] + d_{l_2,0}\big(q_0 - \mathbb{P}[C \le d_{l_2,0}]\big) + d_{l_2,1}q_1 + \cdots + d_{l_2,m}q_m.$$

By (6), we have $\alpha_{l_1} \le \alpha_{l_2}$. □

**Theorem 1.** *For existence of a solution to* (6)*, it is necessary that*

$$\alpha_l \le \min\left\{1 - \frac{1}{r_l}, \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]}\right\} \quad \textit{for all } s_0 \le l \le s, \tag{8}$$

*where*

$$f\big(c_1^*, c_2^*, \ldots, c_m^*\big) = \mathbb{E}\big[C \mid C \le c_1^*\big]\,q_0 + c_1^*q_1 + \cdots + c_m^*q_m.$$

**Proof.** By Lemma 1, the maximum value of the right-hand side of (6) is attained when $d_{l,0} = c_1^*$, $d_{l,1} = c_1^*$, $d_{l,2} = c_2^*$, …, $d_{l,m} = c_m^*$ and equals to $\mathbb{E}[C \mid C \le c_1^*]\,q_0 + c_1^*q_1 + \cdots + c_m^*q_m$. Therefore, we get

$$\alpha_l \le \frac{\mathbb{E}[C \mid C \le c_1^*]\,q_0 + c_1^*q_1 + \cdots + c_m^*q_m}{\mathbb{E}[C]}, \quad s_0 \le l \le s. \tag{9}$$

Since

$$\mathbb{E}\big[C \mid C \le c_1^*\big]\,q_0 + c_1^*q_1 + \cdots + c_m^*q_m < \mathbb{E}\big[C \mid C \le c_1^*\big]\,q_0 + \mathbb{E}\big[C \mid c_1^* < C \le c_2^*\big]\,q_1 + \cdots + \mathbb{E}\big[C \mid C > c_m^*\big]\,q_m = \mathbb{E}[C],$$

the right-hand side of (9) is less than 1. Thus, combining (9) and the inequality $\alpha_l \le 1 - 1/r_l$, which follows immediately from Assumption 1, gives (8). □

**Lemma 2.** *Let condition* (8) *hold. Then for any fixed $l$ such that $s_0 \le l \le s$, we can always choose $d_{l,0}$, $d_{l,1}$, …, $d_{l,m}$ satisfying assertions* (i) *and* (ii) *of Assumption* 2 *such that* (6) *is true.*

The assertion of the lemma is evident.

**Remark 1.** Note only that if

$$\min\left\{1 - \frac{1}{r_l}, \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]}\right\} = \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]},$$

i.e.

$$\alpha_l = \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]},$$

then the combination is unique: $d_{l,0} = c_1^*$, $d_{l,1} = c_1^*$, $d_{l,2} = c_2^*$, …, $d_{l,m} = c_m^*$. Otherwise, if

$$\alpha_l < \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]},$$

then there are infinitely many such combinations.

Next, note that by Lemma 2, we can choose $d_{l,0}$, $d_{l,1}$, …, $d_{l,m}$ for any fixed $l$, but assertion (iii) of Assumption 2 may not hold.

The next theorem shows that we can always replace the bonus–malus system described in Section 2 by the bonus–malus system with varying deductibles.

**Theorem 2.** *There is always a solution to* (6)*, i.e. a combination of* $\alpha_l$, $d_{l,0}$, $d_{l,1}$, $\ldots$, $d_{l,m}$*, where* $s_0 \le l \le s$*, such that Assumptions 1 and 2 hold.*

**Proof.** Consider now two cases.

1) If

$$\min\left\{1 - \frac{1}{r_{s_0}}, \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]}\right\} = \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]},$$

let

$$\alpha_{s_0} = \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]}.$$

By Lemma 2 and Remark 1 applied to $l = s_0$, we get $d_{s_0,0} = c_1^*$, $d_{s_0,1} = c_1^*$, $d_{s_0,2} = c_2^*$, $\ldots$, $d_{s_0,m} = c_m^*$. To satisfy Assumptions 1 and 2, for all $l$ such that $s_0 + 1 \le l \le s$, we set $\alpha_l = \alpha_{s_0}$, $d_{l,0} = d_{s_0,0}$, $d_{l,1} = d_{s_0,1}$, $\ldots$, $d_{l,m} = d_{s_0,m}$, which proves the theorem in the first case.

2) If

$$\min\left\{1 - \frac{1}{r_{s_0}}, \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]}\right\} = 1 - \frac{1}{r_{s_0}},$$

let

$$\alpha_{s_0} = 1 - \frac{1}{r_{s_0}}.$$

By Lemma 2 and Remark 1 applied to $l = s_0$, we can always choose required $d_{s_0,0}$, $d_{s_0,1}$, $\ldots$, $d_{s_0,m}$ and there are infinitely many such combinations. We take any of them. Finally, to satisfy Assumptions 1 and 2, for all $l$ such that $s_0 + 1 \le l \le s$, we also set $\alpha_l = \alpha_{s_0}$, $d_{l,0} = d_{s_0,0}$, $d_{l,1} = d_{s_0,1}$, $\ldots$, $d_{l,m} = d_{s_0,m}$, which proves the theorem in the second case. □

The considerations given above show that the replacement of the bonus–malus system described in Section 2 (without deductibles) by the bonus–malus system with varying deductibles is not unique. So the insurance company can consider different replacements and choose one that seems to be more attractive from the policyholders' point of view.

We now consider two special cases. Another one is considered in Section 4.

**Example 1.** We now suppose that deductibles are applied only to claims of type $m$, i.e. $d_{l,i} = 0$ for all $s_0 \le l \le s$ and $0 \le i \le m - 1$. Therefore, (6) can be rewritten as

$$\alpha_l \mathbb{E}[C] = d_{l,m} q_m. \tag{10}$$

Taking into account Theorem 1, we take any $\alpha_{s_0}$ such that

$$0 < \alpha_{s_0} \le \min\left\{1 - \frac{1}{r_{s_0}}, \frac{c_m^* q_m}{\mathbb{E}[C]}\right\}.$$

By (10), $d_{s_0,m} = \alpha_{s_0} \mathbb{E}[C]/q_m$. To satisfy Assumptions 1 and 2, we set $\alpha_l = \alpha_{s_0}$ and $d_{l,m} = d_{s_0,m}$ for all $s_0 + 1 \le l \le s$. Thus, we get a simple replacement to the bonus–malus system.

**Example 2.** Let $\alpha_l = 1 - 1/r_l$ for all $s_0 \le l \le s$. By Theorem 1, if

$$\frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]} < 1 - \frac{1}{r_{s_0}},$$

then (6) has no suitable solution. So the bonus–malus system cannot be replaced in this way.

## 4 The case where deductibles are applied only to the claims reported by policyholders occupying the highest level in the bonus–malus scale

We now consider the special case where $d_{l,i} = 0$ for all $s_0 \le l \le s-1$ and $0 \le i \le m$. This case is of great interest when the premium relativity for policyholders who are at level $s$ is high enough and varying deductibles are applied to soften the bonus–malus system for such policyholders. To meet the conditions of Assumption 1, we require that $(1-\alpha_s)r_s \ge r_{s-1}$, which gives $\alpha_s \le 1 - r_{s-1}/r_s$. So we can choose any positive $\alpha_s$ such that

$$\alpha_s \le \min\left\{1 - \frac{r_{s-1}}{r_s}, \ \frac{f(c_1^*, c_2^*, \ldots, c_m^*)}{\mathbb{E}[C]}\right\} \tag{11}$$

and then find $d_{s,0}, d_{s,1}, \ldots, d_{s,m}$ from equation (6) applied to $l = s$, i.e. from

$$\begin{aligned}\alpha_s \mathbb{E}[C] = {} & \mathbb{E}[C \mid C \le d_{s,0}]\,\mathbb{P}[C \le d_{s,0}] + d_{s,0}\left(q_0 - \mathbb{P}[C \le d_{s,0}]\right) \\ & + d_{s,1}q_1 + \cdots + d_{s,m}q_m.\end{aligned} \tag{12}$$

If inequality (11) is strict, then there are infinitely many solutions to (12). We now consider two allocation principles for the deductibles.

The *first principle* is when the deductibles are proportional to the average claims of each type. This principle seems to be natural and fair for policyholders, but that is not always possible, which is easily seen from the next theorem.

**Theorem 3.** *Let $\alpha_s$ be such that inequality (11) is strict and set*

$$x_0 = \min\left\{\frac{c_1^*}{\mathbb{E}[C \mid c_1^* < C \le c_2^*]}, \frac{c_2^*}{\mathbb{E}[C \mid c_2^* < C \le c_3^*]}, \ldots, \frac{c_m^*}{\mathbb{E}[C \mid C > c_m^*]}\right\}. \tag{13}$$

*If*

$$\begin{aligned}\alpha_s \mathbb{E}[C] > {} & \mathbb{E}\big[C \mid C \le x_0\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big]\,\mathbb{P}\big[C \le x_0\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big] \\ & + x_0\big(\mathbb{E}\big[C \mid C \le c_1^*\big]\left(q_0 - \mathbb{P}\big[C \le x_0\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big]\right) \\ & + \mathbb{E}\big[C \mid c_1^* < C \le c_2^*\big]\,q_1 + \cdots + \mathbb{E}\big[C \mid C > c_m^*\big]\,q_m\big),\end{aligned} \tag{14}$$

*then we cannot allocate deductibles proportionally to the average claims.*

*Otherwise, if*

$$\begin{aligned}\alpha_s \mathbb{E}[C] \le {} & \mathbb{E}\big[C \mid C \le x_0\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big]\,\mathbb{P}\big[C \le x_0\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big] \\ & + x_0\big(\mathbb{E}\big[C \mid C \le c_1^*\big]\left(q_0 - \mathbb{P}\big[C \le x_0\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big]\right) \\ & + \mathbb{E}\big[C \mid c_1^* < C \le c_2^*\big]\,q_1 + \cdots + \mathbb{E}\big[C \mid C > c_m^*\big]\,q_m\big),\end{aligned} \tag{15}$$

*then that is possible and* (12) *has a unique solution of such kind expressed by*

$$d_{s,0} = x\,\mathbb{E}\big[C \mid C \le c_1^*\big], \qquad d_{s,1} = x\,\mathbb{E}\big[C \mid c_1^* < C \le c_2^*\big], \quad \dots, \\ d_{s,m} = x\,\mathbb{E}\big[C \mid C > c_m^*\big], \tag{16}$$

*where $x$ is a unique positive solution to the equation*

$$\begin{aligned}\alpha_s \mathbb{E}[C] = \mathbb{E}\big[C \mid C \le x\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big]\,\mathbb{P}\big[C \le x\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big] \\ + x\big(\mathbb{E}\big[C \mid C \le c_1^*\big]\big(q_0 - \mathbb{P}\big[C \le x\,\mathbb{E}\big[C \mid C \le c_1^*\big]\big]\big) \\ + \mathbb{E}\big[C \mid c_1^* < C \le c_2^*\big]\,q_1 + \dots + \mathbb{E}\big[C \mid C > c_m^*\big]\,q_m\big).\end{aligned} \tag{17}$$

**Proof.** Let $x$ be a proportionality coefficient, i.e. we have (16). To meet the assertion (i) of Assumption 2, we require that

$$x\,\mathbb{E}\big[C \mid C \le c_1^*\big] \le c_1^*, \qquad x\,\mathbb{E}\big[C \mid c_1^* < C \le c_2^*\big] \le c_1^*, \\ x\,\mathbb{E}\big[C \mid c_2^* < C \le c_3^*\big] \le c_2^*, \quad \dots, \quad x\,\mathbb{E}\big[C \mid C > c_m^*\big] \le c_m^*,$$

which is equivalent to

$$x \le \min\left\{ \frac{c_1^*}{\mathbb{E}[C \mid c_1^* < C \le c_2^*]}, \frac{c_2^*}{\mathbb{E}[C \mid c_2^* < C \le c_3^*]}, \dots, \frac{c_m^*}{\mathbb{E}[C \mid C > c_m^*]} \right\}.$$

So we suppose that $x \in [0, x_0]$, where $x_0$ is given by (13), and substitute (16) into (12), which yields (17).

The left-hand side of (17) is a positive constant. By Lemma 1, the right-hand side of (17) is an increasing function of $x$ on $[0, x_0]$. By the continuity of $F_C$, it is continuous in $x$ on $[0, x_0]$. Moreover, this function is equal to 0 as $x = 0$. Therefore, if (15) holds, then (17) has a unique solution $x \in [0, x_0]$. Otherwise, if (14) is true, then there is no solution $x \in [0, x_0]$ to (17), which completes the proof. □

**Remark 2.** Equation (17) is not solvable analytically in the general case. To find $x$, we should use numerical methods.

The *second principle* implies that large claims are penalized by means of deductibles strictly and small claims are not penalized at all (if that is possible) or at least not so strictly. Anyway (12) must hold. Firstly, we check if it is possible to penalize only claims of type $m$. To this end, we set $d_{s,0} = d_{s,1} = \dots = d_{s,m-1} = 0$, $d_{s,m} = c_m^*$ and substitute this into (12). If

$$\alpha_s \mathbb{E}[C] \le c_m^* q_m,$$

that is possible and the desired allocation is given by

$$d_{s,0} = d_{s,1} = \dots = d_{s,m-1} = 0 \qquad \text{and} \qquad d_{s,m} = \alpha_s \mathbb{E}[C]/q_m.$$

Otherwise, if

$$\alpha_s \mathbb{E}[C] > c_m^* q_m,$$

we also have to penalize at least claims of type $m-1$. We set $d_{s,0} = d_{s,1} = \cdots = d_{s,m-2} = 0$, $d_{s,m-1} = c^*_{m-1}$, $d_{s,m} = c^*_m$ and substitute this into (12). If

$$\alpha_s \mathbb{E}[C] \le c^*_{m-1} q_{m-1} + c^*_m q_m,$$

then the desired allocation is given by

$$d_{s,0} = d_{s,1} = \cdots = d_{s,m-2} = 0,$$
$$d_{s,m-1} = \frac{\alpha_s \mathbb{E}[C] - c^*_m q_m}{q_{m-1}} \quad \text{and} \quad d_{s,m} = c^*_m.$$

Otherwise, we also have to penalize at least claims of type $m-2$. So we set $d_{s,0} = d_{s,1} = \cdots = d_{s,m-3} = 0$, $d_{s,m-2} = c^*_{m-2}$, $d_{s,m-1} = c^*_{m-1}$, $d_{s,m} = c^*_m$, substitute this into (12) and continue in this way until we get the desired allocation. Note that such an allocation is always possible provided that inequality (11) holds.

## 5 Numerical illustrations

We now consider the scale with 4 levels (numbered from 0 to 3) and 4 claim types (numbered from 0 to 3 with probabilities $q_0$, $q_1$, $q_2$ and $q_3$). If no claims are reported during the current year, the policyholder moves one level down. Each claim of type 0 is penalized by one level, each claim of type 1 is penalized by 2 levels, each claim of type 2 or 3 is penalized by 3 levels. Note that penalties for claims of types 2 and 3 are different due to varying deductibles. For instance, if 2 claims of type 0 are reported during the year, then the policyholder moves 2 levels up; if 1 claim of type 0 and 1 claim of type 1 are reported, then the policyholder moves 3 levels up, i.e. goes at the highest level anyway.

The elements of the one-step transition matrix for a policyholder with annual mean claim frequency $\lambda\theta$ and vector of probabilities $\mathbf{q} = (q_0, q_1, q_2, q_3)^T$ are calculated using formula (1). Thus, the one-step transition matrix is given by

$$P(\lambda\theta; \mathbf{q})$$
$$= \begin{pmatrix} e^{-\lambda\theta} & \lambda\theta q_0 e^{-\lambda\theta} & \lambda\theta q_1 e^{-\lambda\theta} + \frac{(\lambda\theta q_0)^2}{2} e^{-\lambda\theta} & 1 - e^{-\lambda\theta} - \lambda\theta(q_0+q_1)e^{-\lambda\theta} - \frac{(\lambda\theta q_0)^2}{2} e^{-\lambda\theta} \\ e^{-\lambda\theta} & 0 & \lambda\theta q_0 e^{-\lambda\theta} & 1 - e^{-\lambda\theta} - \lambda\theta q_0 e^{-\lambda\theta} \\ 0 & e^{-\lambda\theta} & 0 & 1 - e^{-\lambda\theta} \\ 0 & 0 & e^{-\lambda\theta} & 1 - e^{-\lambda\theta} \end{pmatrix}.$$

Next, the stationary probabilities $\pi_l(\lambda\theta; \mathbf{q})$, $0 \le l \le 3$, are calculated using formula (2). A standard computation shows that

$$\pi_0(\lambda\theta; \mathbf{q}) = \frac{e^{-3\lambda\theta}}{\Delta},$$

$$\pi_1(\lambda\theta; \mathbf{q}) = \frac{e^{-2\lambda\theta} - e^{-3\lambda\theta}}{\Delta},$$

$$\pi_2(\lambda\theta; \mathbf{q}) = \frac{e^{-\lambda\theta} - e^{-2\lambda\theta} - \lambda\theta q_0 e^{-3\lambda\theta}}{\Delta},$$

$$\pi_3(\lambda\theta; \mathbf{q}) = \frac{1 - e^{-\lambda\theta} - 2\lambda\theta q_0 e^{-2\lambda\theta} + \lambda\theta(q_0 - q_1)e^{-3\lambda\theta} - \frac{(\lambda\theta q_0)^2}{2} e^{-3\lambda\theta}}{\Delta},$$

where

$$\Delta = 1 - 2\lambda\theta q_0 e^{-2\lambda\theta} - \lambda\theta q_1 e^{-3\lambda\theta} - \frac{(\lambda\theta q_0)^2}{2} e^{-3\lambda\theta}.$$

It is easily seen that

$$e^{-3\lambda\theta} \geq 0, \qquad e^{-2\lambda\theta} - e^{-3\lambda\theta} \geq 0 \quad \text{and} \quad e^{-\lambda\theta} - e^{-2\lambda\theta} - \lambda\theta q_0 e^{-3\lambda\theta} \geq 0$$

for all $\lambda > 0, \theta \geq 0$ and $0 < q_0 < 1$. Moreover, it is evident that $\sum_{l=0}^{3} \pi_l(\lambda\theta; \mathbf{q}) = 1$. So to see that $\pi_l(\lambda\theta; \mathbf{q})$, $0 \leq l \leq 3$, are indeed probabilities, we must show that

$$1 - e^{-\lambda\theta} - 2\lambda\theta q_0 e^{-2\lambda\theta} + \lambda\theta(q_0 - q_1)e^{-3\lambda\theta} - \frac{(\lambda\theta q_0)^2}{2} e^{-3\lambda\theta} \geq 0 \qquad (18)$$

for all $\lambda > 0, \theta \geq 0, q_0 > 0$ and $q_1 > 0$ such that $q_0 + q_1 < 1$.

It is easy to check that the minimum value of the left-hand side of (18) with respect to $q_0 \geq 0$ and $q_1 \geq 0$ such that $q_0 + q_1 \leq 1$ is attained as $q_0 = 1, q_1 = 0$ and equal to

$$1 - e^{-\lambda\theta} - 2\lambda\theta e^{-2\lambda\theta} + \lambda\theta e^{-3\lambda\theta} - \frac{(\lambda\theta)^2}{2} e^{-3\lambda\theta}. \qquad (19)$$

Introduce the function

$$h(y) = 1 - e^{-y} - 2ye^{-2y} + ye^{-3y} - \frac{y^2}{2} e^{-3y}, \quad y \geq 0.$$

Taking the derivative yields

$$h'(y) = e^{-3y}\big(e^y - 1\big)^2 + 2y\big(2e^{-2y} - e^{-3y}\big) + \frac{3y^2}{2} e^{-3y} \geq 0, \quad y \geq 0.$$

Therefore, $h(y)$ is non-decreasing and its minimum value is attained as $y = 0$ and equals to 0. Consequently, the minimum value of (19) is also 0 as $\lambda\theta = 0$, which gives (18).

In this section, we deal with exponentially distributed claim sizes with mean $\mu > 0$, i.e. the distribution function $F_C$ of $C$ is equal to $F_C(y) = 1 - e^{-y/\mu}$, $y \geq 0$. Thus, we have

$$q_0 = 1 - e^{-c_1^*/\mu}, \qquad q_1 = e^{-c_1^*/\mu} - e^{-c_2^*/\mu},$$
$$q_2 = e^{-c_2^*/\mu} - e^{-c_3^*/\mu}, \qquad q_3 = 1 - e^{-c_3^*/\mu};$$

$$\mathbb{E}\big[C \,|\, C \leq c_1^*\big] = \frac{1}{q_0} \int_0^{c_1^*} \frac{y}{\mu} e^{-y/\mu}\,\mathrm{d}y = \frac{\mu(1 - e^{-c_1^*/\mu}) - c_1^* e^{-c_1^*/\mu}}{1 - e^{-c_1^*/\mu}}$$
$$= \mu - \frac{c_1^* e^{-c_1^*/\mu}}{1 - e^{-c_1^*/\mu}},$$

$$\mathbb{E}\big[C \,|\, c_1^* < C \leq c_2^*\big] = \frac{1}{q_1} \int_{c_1^*}^{c_2^*} \frac{y}{\mu} e^{-y/\mu}\,\mathrm{d}y = \frac{(c_1^* + \mu)e^{-c_1^*/\mu} - (c_2^* + \mu)e^{-c_2^*/\mu}}{e^{-c_1^*/\mu} - e^{-c_2^*/\mu}}$$
$$= \mu + \frac{c_1^* e^{-c_1^*/\mu} - c_2^* e^{-c_2^*/\mu}}{e^{-c_1^*/\mu} - e^{-c_2^*/\mu}},$$

$$\mathbb{E}\big[C \mid c_2^* < C \le c_3^*\big] = \frac{1}{q_2}\int_{c_2^*}^{c_3^*} \frac{y}{\mu}\, e^{-y/\mu}\,\mathrm{d}y = \frac{(c_2^*+\mu)e^{-c_2^*/\mu} - (c_3^*+\mu)e^{-c_3^*/\mu}}{e^{-c_2^*/\mu} - e^{-c_3^*/\mu}}$$
$$= \mu + \frac{c_2^* e^{-c_2^*/\mu} - c_3^* e^{-c_3^*/\mu}}{e^{-c_2^*/\mu} - e^{-c_3^*/\mu}},$$

$$\mathbb{E}\big[C \mid C > c_3^*\big] = \frac{1}{q_3}\int_{c_3^*}^{+\infty} \frac{y}{\mu}\, e^{-y/\mu}\,\mathrm{d}y = \frac{(c_3^*+\mu)e^{-c_3^*/\mu}}{e^{-c_3^*/\mu}} = \mu + c_3^*;$$

$$f\big(c_1^*, c_2^*, c_3^*\big) = \mu - \mu e^{-c_1^*/\mu} - c_1^* e^{-c_1^*/\mu} + c_1^*\big(e^{-c_1^*/\mu} - e^{-c_2^*/\mu}\big)$$
$$+ c_2^*\big(e^{-c_2^*/\mu} - e^{-c_3^*/\mu}\big) + c_3^* e^{-c_3^*/\mu}$$
$$= \mu - \mu e^{-c_1^*/\mu} + \big(c_2^* - c_1^*\big)e^{-c_2^*/\mu} + \big(c_3^* - c_2^*\big)e^{-c_3^*/\mu}.$$

In addition, we suppose that $F_\Theta(\theta) = 1 - e^{-\theta}, \theta \ge 0$.

**Example 3.** Let $\lambda = 0.1, \mu = 2, c_1^* = 1, c_2^* = 2, c_3^* = 4$. Then we have $q_0 \approx 0.3935$, $q_1 \approx 0.2387$, $q_2 \approx 0.2325$, $q_3 \approx 0.1353$. The corresponding values of $\pi_l$ and $r_l$, $0 \le l \le 3$, are calculated using formulas (3) and (4), respectively, and are given in Tables 1–9.

We first consider the case when deductibles are applied only to the claims reported by policyholders occupying the highest level in the bonus–malus system. By (11), we have

$$\alpha_3 \le \min\left\{1 - \frac{r_2}{r_3}, \frac{f(c_1^*, c_2^*, c_3^*)}{\mu}\right\} \approx \left\{1 - \frac{1.8899}{2.1844}, \frac{1.425489}{2}\right\}$$
$$\approx \min\{0.1348, 0.7127\} = 0.1348.$$

By (13), we get $x_0 = 0.6667$. Taking $\alpha_3 = 0.05$ and $\alpha_3 = 0.13$ shows that (15) is true in both cases. Hence, we can allocate deductibles proportionally to the average

**Table 1.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied only to claims reported by policyholders occupying the highest level, $\alpha_3 = 0.05$ and the first principle is used

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.05 | 0.4150 | 0.0230 | 0.0730 | 0.1420 | 0.3004 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0 | 0.3780 | 0 | 0 | 0 | 0 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0 | 0.3309 | 0 | 0 | 0 | 0 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 2.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied only to claims reported by policyholders occupying the highest level, $\alpha_3 = 0.13$ and the first principle is used

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.13 | 0.3801 | 0.0598 | 0.1903 | 0.3699 | 0.7827 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0 | 0.3780 | 0 | 0 | 0 | 0 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0 | 0.3309 | 0 | 0 | 0 | 0 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 3.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied only to claims reported by policyholders occupying the highest level, $\alpha_3 = 0.05$ and the second principle is used

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.05 | 0.4150 | 0 | 0 | 0 | 0.7389 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0 | 0.3780 | 0 | 0 | 0 | 0 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0 | 0.3309 | 0 | 0 | 0 | 0 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 4.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied only to claims reported by policyholders occupying the highest level, $\alpha_3 = 0.13$ and the second principle is used

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.05 | 0.3801 | 0 | 0 | 0 | 1.9212 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0 | 0.3780 | 0 | 0 | 0 | 0 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0 | 0.3309 | 0 | 0 | 0 | 0 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 5.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied only to claims of type 3

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.24 | 0.3320 | 0 | 0 | 0 | 3.5467 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0.13 | 0.3288 | 0 | 0 | 0 | 1.9212 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0.06 | 0.3110 | 0 | 0 | 0 | 0.8867 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 6.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied only to claims of type 3. Larger values of $\alpha_l$

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.26 | 0.3233 | 0 | 0 | 0 | 3.8423 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0.25 | 0.2835 | 0 | 0 | 0 | 3.6945 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0.24 | 0.2514 | 0 | 0 | 0 | 3.5467 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

claims. By (17), the proportionality coefficient $x = 0.050066$ if $\alpha_3 = 0.05$ and $x = 0.130443$ if $\alpha_3 = 0.13$. The corresponding values of deductibles are calculated using (16) and given in Tables 1 and 2.

If we apply the second principle considered in Section 4, we get $\alpha_3 \mu \le c_3^* q_3$ for both values of $\alpha_3$. The desired allocation is presented in Tables 3 and 4.

Tables 5 and 6 give examples of bonus–malus system in the case where deductibles are applied only to claims of type 3. In Table 6, we take larger values of $\alpha_l$, so we get higher values of the deductibles.

If we also apply deductibles to claims of type 2 for the same values of $\alpha_l$, the values of deductibles $d_{l,3}$ are not so high (see Table 7). Moreover, in this case, we can take larger values of $\alpha_l$ such that Assumptions 1 and 2 hold (see Table 8).

Table 9 presents an example of bonus–malus system in the case where deductibles are applied to claims of types 1, 2 and 3. On the one hand, policyholders who are in the

**Table 7.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied to claims of types 2 and 3

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.26 | 0.3233 | 0 | 0 | 1.1 | 1.9522 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0.25 | 0.2835 | 0 | 0 | 1.1 | 1.8044 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0.24 | 0.2514 | 0 | 0 | 1.1 | 1.6566 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 8.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied to claims of types 2 and 3. Larger values of $\alpha_l$

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.45 | 0.2403 | 0 | 0 | 1.7 | 3.7291 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0.40 | 0.2268 | 0 | 0 | 1.6 | 3.1620 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0.35 | 0.2151 | 0 | 0 | 1.5 | 2.5949 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 9.** Bonus–malus system with varying deductibles for Example 3 where deductibles are applied to claims of types 1, 2 and 3

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0508 | 2.1844 | 0.4369 | 0.45 | 0.2403 | 0 | 0.7 | 1.5 | 2.8383 |
| 2 | 0.0591 | 1.8899 | 0.3780 | 0.40 | 0.2268 | 0 | 0.5 | 1.4 | 2.6239 |
| 1 | 0.0716 | 1.6543 | 0.3309 | 0.35 | 0.2151 | 0 | 0.3 | 1.3 | 2.4096 |
| 0 | 0.8185 | 0.8050 | 0.1610 | 0 | 0.1610 | 0 | 0 | 0 | 0 |

**Table 10.** Bonus–malus system with varying deductibles for Example 4 where deductibles are applied to claims of types 2 and 3

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0653 | 2.0731 | 0.4146 | 0.20 | 0.3317 | 0 | 0 | 0.30 | 1.2544 |
| 2 | 0.0717 | 1.7925 | 0.3585 | 0.15 | 0.3047 | 0 | 0 | 0.25 | 0.9102 |
| 1 | 0.0679 | 1.6263 | 0.3253 | 0.10 | 0.2927 | 0 | 0 | 0.20 | 0.5659 |
| 0 | 0.7951 | 0.7869 | 0.1574 | 0 | 0.1574 | 0 | 0 | 0 | 0 |

**Table 11.** Bonus–malus system with varying deductibles for Example 4 where deductibles are applied to claims of types 1, 2 and 3

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0653 | 2.0731 | 0.4146 | 0.24 | 0.3151 | 0 | 0.10 | 0.60 | 1.0847 |
| 2 | 0.0717 | 1.7925 | 0.3585 | 0.22 | 0.2796 | 0 | 0.10 | 0.55 | 0.9838 |
| 1 | 0.0679 | 1.6263 | 0.3253 | 0.20 | 0.2602 | 0 | 0.05 | 0.50 | 0.9461 |
| 0 | 0.7951 | 0.7869 | 0.1574 | 0 | 0.1574 | 0 | 0 | 0 | 0 |

malus zone pay not such high premiums. On the other hand, the corresponding values of deductibles are moderate. This bonus–malus system seems to be more attractive.

**Example 4.** Let $\lambda = 0.1$, $\mu = 2$, $c_1^* = 0.3$, $c_2^* = 1.2$, $c_3^* = 2.8$. Then we have $q_0 \approx 0.1393$, $q_1 \approx 0.3119$, $q_2 \approx 0.3022$, $q_3 \approx 0.2466$. The corresponding values of $\pi_l$ and $r_l$ as well as examples of bonus–malus systems with varying deductibles are given in Tables 10–12.

**Table 12.** Bonus–malus system with varying deductibles for Example 4 where deductibles are applied to claims of types 1, 2 and 3. Larger values of $\alpha_l$

| $l$ | $\pi_l$ | $r_l$ | $\lambda r_l \mathbb{E}[C]$ | $\alpha_l$ | $(1-\alpha_l)\lambda r_l \mathbb{E}[C]$ | $d_{l,0}$ | $d_{l,1}$ | $d_{l,2}$ | $d_{l,3}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.0653 | 2.0731 | 0.4146 | 0.45 | 0.2280 | 0 | 0.20 | 0.9 | 2.2937 |
| 2 | 0.0717 | 1.7925 | 0.3585 | 0.40 | 0.2151 | 0 | 0.15 | 0.8 | 2.0740 |
| 1 | 0.0679 | 1.6263 | 0.3253 | 0.35 | 0.2114 | 0 | 0.10 | 0.7 | 1.8543 |
| 0 | 0.7951 | 0.7869 | 0.1574 | 0 | 0.1574 | 0 | 0 | 0 | 0 |

## 6 Conclusion

The bonus–malus systems with different claim types and varying deductibles eliminate both drawbacks of the traditional bonus–malus systems mentioned in Section 1 and present a number of advantages, namely:

- Policyholders reporting small and large claims are not penalized in the same way. This helps to avoid or at least decrease bonus hunger.
- Policyholders will do all their best to prevent or at least decrease the losses.
- Even if a policyholder leaves the company after a claim, he has to pay for the deductible.
- Relative premiums and amounts of deductibles may be tuned in an optimal way in order to attract policyholders.

Section 5 gives a few examples of such bonus–malus systems. The numerical illustrations show that use of both penalty types (premium relativities and varying deductibles) in this way seems indeed attractive and fair for policyholders. On the one hand, policyholders who are in the malus zone pay not such high premiums. On the other hand, the corresponding values of deductibles are moderate. In addition, it is fair that only policyholders reporting (large) claims are subject to further penalties, i.e. deductibles.

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