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# The self-normalized Donsker theorem revisited

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**Abstract** We extend the Poincaré–Borel lemma to a weak approximation of a Brownian motion via simple functionals of uniform distributions on n-spheres in the Skorokhod space $D([0,1])$. This approach is used to simplify the proof of the self-normalized Donsker theorem in Csörgő et al. (2003). Some notes on spheres with respect to $\ell_p$-norms are given.

**Keywords** Poincaré–Borel lemma, Brownian motion, Donsker theorem, self-normalized sums

**2010 MSC** 60F05, 60F17

## 1 Introduction

Let $\mathcal{S}^{n-1}(d) = \{x \in \mathbb{R}^n : \|x\| = d\}$ be the $(n-1)$-sphere with radius $d$, where $\|\cdot\|$ denotes the Euclidean norm. The uniform measure on the unit sphere $\mathcal{S}^{n-1} := \mathcal{S}^{n-1}(1)$ can be characterized as

$$\mu_{S,n} \overset{d}{=} \frac{(X_1,\dots,X_n)}{\|(X_1,\dots,X_n)\|}, \tag{1}$$

where $(X_1,\dots,X_n)$ is a standard $n$-dimensional normal random variable.

The celebrated Poincaré–Borel lemma is the classical result on the approximation of a Gaussian distribution by projections of the uniform measure on $\mathcal{S}^{n-1}(\sqrt{n})$ as $n$ tends to infinity: Let $n \geq m$ and $\pi_{n,m} : \mathbb{R}^n \to \mathbb{R}^m$ be the natural projection. The uniform measure on the sphere $\mathcal{S}^{n-1}(\sqrt{n})$ is given by $\sqrt{n}\,\mu_{S,n}$. Then, for every fixed $m \in \mathbb{N}$,

$$\sqrt{n}\mu_{S,n} \circ \pi_{n,m}^{-1}$$

converges in distribution to a standard $m$-dimensional normal distribution as $n$ tends to infinity, cf. [11, Proposition 6.1]. Following the historical notes in [6, Section 6] on the earliest reference to this result by Émile Borel, we acquire the usual practice to speak about the Poincaré–Borel lemma.

Among other fields, this convergence stimulated the development of the infinite-dimensional functional analysis (cf. [12]) as well as the concentration of measure theory (cf. [10, Section 1.1]).

In particular, it inspired to consider connections of the Wiener measure and the uniform measure on an infinite-dimensional sphere [21]. Such a Donsker-type result is firstly proved in [4] by nonstandard methods. For the illustration, we make use of the notations in [7], where this result is used for statistical analysis of measures on high-dimensional unit spheres. Define the functional

$$Q_{n,2} : \mathcal{S}^{n-1} \to C\big([0,1]\big), \qquad (x_1, \dots, x_n) \mapsto \big(Q_{n,2}(t)\big)_{t\in[0,1]},$$

such that

$$Q_{n,2}(k/n) := \frac{\sum_{i=1}^{k} x_i}{\|(x_1, \dots, x_n)\|},$$

for $k \in \{0, \dots, n\}$ and is linearly interpolated elsewhere. Then [4, Theorem 2.4] gives that the sequence of processes

$$\mu_{S,n} \circ Q_{n,2}^{-1}$$

converges weakly to a Brownian motion $W := (W_t)_{t\in[0,1]}$ in the space of continuous functions $C([0,1])$ as $n$ tends to infinity. The first proof without nonstandard methods in $C([0,1])$ and in the Skorohod space $D([0,1])$ is given in [17].

In this note, we present a very simple proof of the càdlàg version of this Poincaré–Borel lemma for Brownian motion. This is the content of Section 2.

Some remarks on such Donsker-type convergence results on spheres with respect to $\ell_p$-norms are collected in Section 3.

In fact, our simple approach can be used to simplify the proof of the main result in [3] as well. This is presented in Section 4.

## 2 Poincaré–Borel lemma for Brownian motion

Suppose $X_1, X_2, \dots$ is a sequence of i.i.d. standard normal random variables. Then $(X_1, \dots, X_n)$ has a standard $n$-dimensional normal distribution. We define the processes with càdlàg paths

$$Z^n = \left( Z_t^n := \frac{\sum_{i=1}^{\lfloor nt \rfloor} X_i}{\|(X_1, \dots, X_n)\|} \right)_{t\in[0,1]}.$$

Thus, $Z^n$ is equivalent to $\mu_{S,n} \circ \overline{Q}_{n,2}^{-1}$ for the functional

$$\overline{Q}_{n,2} : \mathcal{S}^{n-1} \to D\big([0,1]\big), \qquad (x_1, \dots, x_n) \mapsto \left( \overline{Q}_{n,2}(t) = \frac{\sum_{i=1}^{\lfloor nt \rfloor} x_i}{\|(x_1, \dots, x_n)\|} \right)_{t\in[0,1]},$$

and therefore it is a relatively simple computation from the uniform distribution on the $n$-sphere. Then the following extension of the Poincaré–Borel lemma is true:

**Theorem 1.** *The sequence* $(Z^n)_{n\in\mathbb{N}}$ *converges weakly in the Skorokhod space* $D([0,1])$ *to a standard Brownian motion* $W$ *as* $n$ *tends to infinity.*

**Proof.** As the distribution of the random vector in (1) is exactly the uniform measure $\mu_{S,n}$, the proof of the convergence of finite-dimensional distributions is in line with the classical Poincaré–Borel lemma: by the law of large numbers, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 1$ in probability. Hence, by the continuous mapping theorem, $\sqrt{n}/(\|(X_1,\dots,X_n)\|) \to 1$ in probability, and, by Donsker's theorem and Slutsky's theorem, we conclude the convergence of finite-dimensional distributions.

For the tightness we consider the increments of the process $Z^n$ and make use of a standard criterion. For all $s \le t$ in $[0,1]$, we denote

$$\left(Z_t^n - Z_s^n\right)^2 = \frac{\sum_{\lfloor ns\rfloor < i \le \lfloor nt\rfloor} X_i^2}{\sum_{i\le n} X_i^2} + \frac{\sum_{\lfloor ns\rfloor < i \ne j \le \lfloor nt\rfloor} X_i X_j}{\sum_{i\le n} X_i^2} =: I_1^{t,s} + I_2^{t,s}. \tag{2}$$

Due to the symmetry of the standard $n$-dimensional normally distributed vector $(X_1,\dots,X_n)$, for all pairwise different $i, j, k, l$, we observe

$$\mathbb{E}\left[\frac{X_i X_j X_k X_l}{(\sum_{i\le n} X_i^2)^2}\right] = \mathbb{E}\left[\frac{X_i^2 X_j X_k}{(\sum_{i\le n} X_i^2)^2}\right] = 0. \tag{3}$$

Let $s \le u \le t$ in $[0,1]$. Thus via (3), we conclude

$$\mathbb{E}\left[I_1^{t,u} I_2^{u,s}\right] = 0, \qquad \mathbb{E}\left[I_2^{t,u} I_1^{u,s}\right] = 0, \qquad \mathbb{E}\left[I_2^{t,u} I_2^{u,s}\right] = 0,$$

and therefore

$$\mathbb{E}\left[\left(Z_t^n - Z_u^n\right)^2\left(Z_u^n - Z_s^n\right)^2\right] = \mathbb{E}\left[I_1^{t,u} I_1^{u,s}\right].$$

We denote for shorthand $m_1 := \lfloor nt\rfloor - \lfloor nu\rfloor$, $m_2 := \lfloor nu\rfloor - \lfloor ns\rfloor$ and $m_3 := n - (\lfloor nt\rfloor - \lfloor ns\rfloor)$. Then we observe

$$I_1^{t,u} I_1^{u,s} = \frac{\chi_{m_1}^2 \chi_{m_2}^2}{(\chi_{m_1}^2 + \chi_{m_2}^2 + \chi_{m_3}^2)^2} = \frac{\frac{1}{2}((\chi_{m_1}^2 + \chi_{m_2}^2)^2 - (\chi_{m_1}^2)^2 - (\chi_{m_2}^2)^2)}{(\chi_{m_1}^2 + \chi_{m_2}^2 + \chi_{m_3}^2)^2},$$

for pairwise independent chi-squared random variables $\chi_m^2$ with $m$ degrees of freedom. We recall that $\frac{\chi_m^2}{\chi_m^2 + \chi_k^2}$ is Beta$(m/2, k/2)$-distributed with

$$\mathbb{E}\left[\left(\frac{\chi_m^2}{\chi_m^2 + \chi_k^2}\right)^2\right] = \left(\frac{m+2}{m+k+2}\right)\left(\frac{m}{m+k}\right). \tag{4}$$

Hence a computation via (4) yields

$$\begin{aligned}\mathbb{E}\left[I_1^{t,u} I_1^{u,s}\right] &= \frac{m_1 m_2}{(m_1+m_2+m_3+2)(m_1+m_2+m_3)} \\ &\le \left(\frac{m_1}{m_1+m_2+m_3}\right)\left(\frac{m_2}{m_1+m_2+m_3}\right),\end{aligned}$$

and therefore

$$\mathbb{E}\big[\big(Z_t^n - Z_u^n\big)^2\big(Z_u^n - Z_s^n\big)^2\big] \leq \left(\frac{\lfloor nt\rfloor - \lfloor nu\rfloor}{n}\right)\left(\frac{\lfloor nu\rfloor - \lfloor ns\rfloor}{n}\right)$$
$$\leq \left(\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}\right)^2.$$

Thus the well-known criterion [1, Theorem 15.6] (cp. Remark 1 in [15]) implies the tightness of $Z^n$. □

**Remark 2.** (i) The heuristic connection of the Wiener measure and the uniform measure on an infinite-dimensional sphere goes back to Norbert Wiener's study of the *differential space*, [21]. The first informal presentation of Theorem 1 and further historical notes can be found in [12]. The first rigorous proof is given in Section 2 of [4]. However, the authors make use of nonstandard analysis and the functional $Q_{n,2}$. To the best of our knowledge, the first proof of Theorem 1 is [17]. In contrast, our proof is based on the pretty decoupling in the tightness argument. Moreover, this approach is extended in Section 4 to a simpler proof of Theorem 1 in [3].

(ii) According to the historical comments in [20, Section 2.2], the Poincaré–Borel lemma could be also attributed to Maxwell and Mehler.

## 3 $\ell_p^n$-spheres

In this section, we consider uniform measures on $\ell_p^n$-spheres and prove that the limit in Theorem 1 is the only case such that a simple $\overline{Q}$-type pathwise functional leads to a nontrivial limit (Theorem 5).

Furthermore, we present random variables living on $\ell_p^n$-spheres, with a similar characterization for a fractional Brownian motion (Theorem 6).

Concerning the $\ell_p^n$ norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \in [1, \infty)$ and defining the $\ell_p^n$ unit sphere

$$\mathcal{S}_p^{n-1} := \big\{x \in \mathbb{R}^n : \|x\|_p = 1\big\},$$

the uniform measure $\mu_{S,n,p}$ on $\mathcal{S}_p^{n-1}$ is characterized similarly to the uniform measure on the Euclidean unit sphere by independent results in [18, Lemma 1] and [16, Lemma 3.1]:

**Proposition 3.** *Suppose $X, X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with density*

$$f(x) = \frac{\exp(-|x|^p/p)}{2p^{1/p}\Gamma(1+1/p)}.$$

*Then*

$$\mu_{S,n,p} \overset{d}{=} \frac{(X_1, \ldots, X_n)}{\|(X_1, \ldots, X_n)\|_p}.$$

**Remark 4.** (i) We notice that the uniform measure on the $\ell_p^n$-sphere equals the surface measure only in the cases $p \in \{1, 2, \infty\}$, see e.g. [16, Section 3] or the interesting study of the total variation distance of these measures for $p \geq 1$ in [14].

(ii) In particular, we have a counterpart of the classical Poincaré–Borel lemma for finite-dimensional distributions: For every fixed $m \in \mathbb{N}$,

$$n^{1/p} \mu_{S,n,p} \circ \pi_{n,m}^{-1}$$

converges in distribution to the random vector $(X_1, \dots, X_m)$ as $n$ tends to infinity. This follows immediately from $\mathbb{E}[|X|^p] = 1$ and the law of large numbers, cf. [11, Proposition 6.1] or the finite-dimensional convergence in Theorem 1.

Similarly to the characterization of the central limit theorem, cp. [9, Theorem 4.23], but in contrast to the convergence of the projection on a finite number of coordinates in Remark 4, we have a uniqueness result for the processes constructed according to the $\overline{Q}$-type pathwise functionals.

In the following we denote the convergence in distribution by $\xrightarrow{d}$ and the almost sure convergence by $\xrightarrow{a.s.}$.

**Theorem 5.** *Suppose $p \geq 1$ and denote*

$$\overline{Q}_{n,p} : (x_1, \dots, x_n) \mapsto \left( \frac{\sum_{i=1}^{\lfloor nt \rfloor} x_i}{\|(x_1, \dots, x_n)\|_p} \right)_{t \in [0,1]}.$$

*Then, in the Skorokhod space $D([0,1])$, as $n$ tends to infinity:*

$$\mu_{S,n,p} \circ \overline{Q}_{n,p}^{-1} \begin{cases} \xrightarrow{a.s.} 0, & p < 2, \\ \xrightarrow{d} W, & p = 2, \\ \text{is divergent}, & p > 2. \end{cases}$$

**Proof.** The strong law of large numbers [9, Theorem 4.23] implies that $n^{1/p}/\|(X_1, \dots, X_n)\|_p \to 1$ almost surely for all $p \geq 1$. Moreover, for $p < 2$, it gives as well that $\frac{1}{n^{1/p}} \sum_{i=1}^{\lfloor nt \rfloor} X_i \to 0$ almost surely for all $t \in [0,1]$. Thanks to Proposition 3, we have

$$\mu_{S,n,p} \circ \overline{Q}_{n,p}^{-1} \stackrel{d}{=} \frac{n^{1/p}}{\|(X_1, \dots, X_n)\|_p} \left( n^{-1/p} \sum_{i=1}^{\lfloor n \cdot \rfloor} X_i \right).$$

Thus we conclude via $n^{-1/p} = n^{-1/2} n^{(p-2)/2p}$, Donsker's theorem and Slutsky's theorem. □

However, the $\ell_p^n$ spheres can be involved in another convergence result. The fractional Brownian motion $B^H = (B_t^H)_{t \geq 0}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with the covariance $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. We refer to [13] for further information on this generalization of the Brownian motion beyond semimartingales. In particular, there is the following random walk approximation ([19, Theorem 2.1] or [13, Lemma 1.15.9]): Let $\{X_i\}_{i \geq 1}$ be a stationary Gaussian sequence with $\mathbb{E}[X_i] = 0$ and correlations

$$\sum_{i,j=1}^{n} \mathbb{E}[X_i X_j] \sim n^{2H} L(n),$$

as $n$ tends to infinity for a slowly varying function $L$. Then $\frac{1}{n^{2H}L(n)}\sum_{i=1}^{\lfloor nt\rfloor} X_i$ converges weakly in the Skorohod space $D([0,1])$ towards a fractional Brownian motion with Hurst parameter $H$. For simplification let $X_i = B_i^H - B_{i-1}^H$, $i \in \mathbb{N}$, be the correlated increments of the fractional Brownian motion $B^H$. The stationarity and the ergodic theorem imply, for $p > 0$ and the constant $c_H := \mathbb{E}[|B_1^H|^{1/H}]$, that

$$\left(\left\|(X_1,\ldots,X_n)\right\|_p/n^H\right)^p = n^{-Hp}\sum_{i=1}^n |X_i|^p \overset{a.s.}{\to} \begin{cases} 0, & p > 1/H, \\ c_H & p = 1/H, \\ +\infty, & p < 1/H, \end{cases} \tag{5}$$

(see e.g. [13, Eq. (1.18.3)]). With this at hand, we obtain a similar uniqueness result:

**Theorem 6.** *Let $X_i = B_i^H - B_{i-1}^H$, $i \in \mathbb{N}$, be the increments of a fractional Brownian motion $B^H$. Then, in the Skorokhod space $D([0,1])$, as $n$ tends to infinity:*

$$\overline{Q}_{n,p}(X_1,\ldots,X_n) = \left(\frac{\sum_{i=1}^{\lfloor nt\rfloor} X_i}{\|(X_1,\ldots,X_n)\|_p}\right)_{t\in[0,1]} \begin{cases} \overset{a.s.}{\to} 0, & p < 1/H, \\ \overset{d}{\to} B^H/c_H^H, & p = 1/H, \\ \text{is divergent}, & p > 1/H. \end{cases}$$

**Proof.** Taqqu's limit theorem implies, for all $H \in (0,1)$,

$$\left(n^{-H}\sum_{i=1}^{\lfloor nt\rfloor} X_i\right)_{t\in[0,1]} \overset{d}{\to} B^H$$

in the Skorokhod space $D([0,1])$. Then, thanks to (5), we conclude as in Theorem 5. □

**Remark 7.** Due to the different correlations between the random variables $X_i$ in Theorem 6, there is no symmetric and trivial sequence of measures $\hat{\mu}_{S,n,p}$ on the $\ell_p^n$-spheres and some simple $\overline{Q}_{n,p}$-type pathwise functionals, which represent the distributions of $\overline{Q}_{n,p}(X_1,\ldots,X_n)$. However, it would be interesting, whether some uniform or surface measures on geometric objects in combination with simple $\overline{Q}_p$-type pathwise functionals allow similar Donsker-type theorems for fractional Brownian motion or other Gaussian processes?

## 4 The self-normalized Donsker theorem

Suppose $X, X_1, X_2, \ldots$ is a sequence of i.i.d. nondegenerate random variables and we denote for all $n \in \mathbb{N}$,

$$S_n := \sum_{i=1}^n X_i, \qquad V_n^2 := \sum_{i=1}^n X_i^2.$$

Limit theorems for self-normalized sums $S_n/V_n$ play an important role in statistics, see e.g. [8], and have been extensively studied during the last decades, cf. the monograph on self-normalizes processes [5].

In [3], the following invariance principle for self-normalized sum processes is established.

**Theorem 8** (Theorem 1 in [3]). *Assume the notations above and denote*

$$Z_t^n := S_{\lfloor nt \rfloor}/V_n.$$

*Then the following assertions, with n tending to infinity, are equivalent:*

*(a)* $E[X] = 0$ *and* $X$ *is in the domain of attraction of the normal law (i.e. there exists a sequence* $(b_n)_{n\geq 1}$ *with* $S_n/b_n \xrightarrow{d} \mathcal{N}(0,1)$*).*

*(b) For all* $t_0 \in (0,1]$, $Z_{t_0}^n \xrightarrow{d} \mathcal{N}(0,t_0)$.

*(c)* $(Z_t^n)_{t\in[0,1]}$ *converges weakly to* $(W_t)_{t\in[0,1]}$ *on* $(D([0,1]),\rho)$, *where* $\rho$ *denotes the uniform topology.*

*(d) On an appropriate joint probability space, the following is valid:*

$$\sup_{t\in[0,1]} \left| Z_t^n - W(nt)/\sqrt{n} \right| = o_P(1).$$

**Remark 9.** The equivalence of $(a)$ and $(b)$ is the celebrated result [8, Theorem 3]. Since the implications $(d) \Rightarrow (c) \Rightarrow (b)$ are trivial, the proof in [3] is completed by showing $(a) \Rightarrow (d)$.

Thanks to a tightness argument as in the proof of Theorem 1, we obtain a simpler alternative for the proof.

**Proof of Theorem 8.** As stated in the remark, we already know that $(d) \Rightarrow (c) \Rightarrow (b) \Leftrightarrow (a)$. We denote

$(c_0)$ $(Z_t^n)_{t\in[0,1]}$ converges weakly to $(W_t)_{t\in[0,1]}$ on the Skorokhod space $D([0,1])$.

By the continuity of the paths of the Brownian motion and [1, Section 18], we obtain the equivalence $(c) \Leftrightarrow (c_0)$. We denote by $d_0$ the Skorokhod metric on $D([0,1])$ which makes it a Polish space. The Skorokhod–Dudley Theorem [9, Theorem 4.30] and $(c_0)$ imply

$$d_0\big((Z_t^n)_{t\in[0,1]}, (W_t)_{t\in[0,1]}\big) \to 0,$$

almost surely on an appropriate probability space. Since the uniform topology is finer than the Skorokhod topology ([1, Section 18]), we conclude assertion $(d)$. Thus it remains to prove $(a) \Rightarrow (c_0)$. Firstly we consider finite-dimensional distributions. Due to [8, Lemma 3.2], the sequence $(b_n)_{n\in\mathbb{N}}$ with $S_n/b_n \xrightarrow{d} \mathcal{N}(0,1)$ fulfills $V_n/b_n \to 1$ in probability and $b_n = \sqrt{n}L(n)$ for some slowly varying at infinity function $L$. The continuous mapping theorem implies $b_n/V_n \to 1$ in probability. Take arbitrary $N \in \mathbb{N}$, $a_1,\dots,a_N \in \mathbb{R}$ and $t_1,\dots,t_N \in [0,1]$. Without loss of generality, we assume $t_1 < \cdots < t_N$ and denote $t_0 := 0$ and $t_{N+1} := 1$. Then, by the independence of the random variables $S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor}$, $i = 1,\dots,N+1$, for every fixed $n \in \mathbb{N}$, Lévy's continuity theorem and the normality of the random vector $(Y_1,\dots,Y_{N+1})$, we obtain

$$\left( \frac{S_{\lfloor nt_1 \rfloor} - S_{\lfloor nt_0 \rfloor}}{\sqrt{(\lfloor nt_1 \rfloor)}}, \dots, \frac{S_{\lfloor nt_{N+1} \rfloor} - S_{\lfloor nt_N \rfloor}}{\sqrt{(\lfloor nt_{N+1} \rfloor - \lfloor nt_N \rfloor)}} \right) \xrightarrow{d} (Y_1,\dots,Y_{N+1}),$$

as $n$ tends to infinity. As the sequence $(b_n)_{n\in\mathbb{N}}$ is regularly varying with exponent $1/2$, it is easily seen that

$$\frac{b_{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor}}{b_n}\to\sqrt{t_i-t_{i-1}}.$$

Via the continuous mapping theorem, we conclude

$$\sum_i a_i\frac{S_{\lfloor nt_i\rfloor}}{b_n}=\sum_{i=1}^{N+1}\frac{(\sum_{j\le i}a_j)(b_{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor})}{b_n}\left(\frac{S_{\lfloor nt_i\rfloor}-S_{\lfloor nt_{i-1}\rfloor}}{b_{\lfloor nt_i\rfloor-\lfloor nt_{i-1}\rfloor}}\right)$$
$$\xrightarrow{d}\sum_{i=1}^{N+1}\Big(\sum_{j\le i}a_j\Big)\sqrt{t_i-t_{i-1}}Y_i\stackrel{d}{=}\sum_{i=1}^{N+1}a_iW_{t_i}.$$

Slutsky's theorem implies

$$\sum_{i=1}^{N+1}a_iZ^n_{t_i}=\left(\frac{b_n}{V_n}\right)\left(\sum_i a_i\frac{S_{\lfloor nt_i\rfloor}}{b_n}\right)\xrightarrow{d}\sum_{i=1}^{N+1}a_iW_{t_i},$$

what means the convergence of finite-dimensional distributions.

The tightness follows again by the criterion [1, Theorem 15.6]. By the identical distribution, for all $m\le n$, we have

$$\mathbb{E}\left[\left(\frac{\sum_{i\le m}X_i^2}{\sum_{i\le n}X_i^2}\right)^2\right]=\mathbb{E}\left[\frac{mX_1^4}{(\sum_{i\le n}X_i^2)^2}\right]+\mathbb{E}\left[\frac{m(m-1)X_1^2X_2^2}{(\sum_{i\le n}X_i^2)^2}\right].\tag{6}$$

Thanks to the value 1 on the left hand side in (6) for $m=n$, we conclude

$$0\le\mathbb{E}\left[\frac{X_1^2X_2^2}{(\sum_{i\le n}X_i^2)^2}\right]\le\frac{1}{n(n-1)}.$$

In contrast to (3), for possibly nonsymmetric random variables, the Cauchy–Schwarz inequality and [8, (3.10)] yields a constant $c_X<\infty$ such that for every $r\in\{2,3,4\}$,

$$\max_{\substack{i,j,k,l\le n\\ |\{i,j,k,l\}|=r}}\mathbb{E}\left[\frac{|X_iX_jX_kX_l|}{(\sum_{i\le n}X_i^2)^2}\right]\le c_Xn^{-r}.\tag{7}$$

Applying the estimates in (7) on the terms in (2) gives that

$$\max_{i,j\in\{1,2\}}\mathbb{E}\big[I_i^{t,u}I_j^{u,s}\big]\le c_X\left(\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right)^2.\tag{8}$$

Hence, we obtain

$$\mathbb{E}\big[(Z^n_t-Z^n_u)^2(Z^n_u-Z^n_s)^2\big]=\mathbb{E}\big[(I_1^{t,u}+I_2^{t,u})(I_1^{u,s}+I_2^{u,s})\big]\le 4c_X\left(\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right)^2,$$

and the proof concludes as in Theorem 1. □

**Remark 10.** (i) By the same reasoning, we obtain Theorem 5 for the sequence of i.i.d. variables $X, X_1, X_2, \ldots$ such that Theorem 8 ($a$) is fulfilled.

(ii) In [2], a similar counterpart of Theorem 8 for $\alpha$-stable Lévy processes is established. An interesting question would be on a uniqueness result similar to Theorem 5.

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