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Asymptotic behavior of functionals of the solutions to inhomogeneous Itô stochastic differential equations with nonregular dependence on parameter

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Abstract The asymptotic behavior, as $T \to \infty$, of some functionals of the form $I_T(t) = F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s))\,dW_T(s)$, $t \geq 0$ is studied. Here $\xi_T(t)$ is the solution to the time-inhomogeneous Itô stochastic differential equation

$$d\xi_T(t) = a_T\big(t, \xi_T(t)\big)\,dt + dW_T(t), \quad t \geq 0,\ \xi_T(0) = x_0,$$

$T > 0$ is a parameter, $a_T(t,x), x \in \mathbb{R}$ are measurable functions, $|a_T(t,x)| \leq C_T$ for all $x \in \mathbb{R}$ and $t \geq 0$, $W_T(t)$ are standard Wiener processes, $F_T(x), x \in \mathbb{R}$ are continuous functions, $g_T(x), x \in \mathbb{R}$ are measurable locally bounded functions, and everything is real-valued. The explicit form of the limiting processes for $I_T(t)$ is established under nonregular dependence of $a_T(t,x)$ and $g_T(x)$ on the parameter T.

Keywords Diffusion-type processes, asymptotic behavior of functionals, nonregular dependence on the parameter
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1 Introduction

Consider the time-inhomogeneous Itô stochastic differential equation

$$d\xi_T(t) = a_T\big(t, \xi_T(t)\big)\,dt + dW_T(t), \quad t \geq 0,\ \xi_T(0) = x_0, \tag{1}$$

where $T > 0$ is a parameter, $a_T(t,x)$, $x \in \mathbb{R}$ are real-valued measurable functions such that $|a_T(t,x)| \leq L_T$ for all (t,x) and some family of constants $L_T > 0$, and $W_T = \{W_T(t), t \geq 0\}$, $T > 0$ is a family of standard Wiener processes defined on a complete probability space $(\Omega, \Im, \mathsf{P})$.

It is known from Theorem 4 in [12] that for any $T > 0$ and $x_0 \in \mathbb{R}$ equation (1) possesses a unique strong solution $\xi_T = \{\xi_T(t), t \geq 0\}$.

In this paper, we study the weak convergence, as $T \to \infty$, of the processes $I_T(t) = F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s))\,dW_T(s)$, where $\xi_T(t)$ is the solution to stochastic differential equation (1), $F_T(x)$ is a family of continuous real-valued functions, $g_T(x)$ is a family of measurable, locally bounded real-valued functions. All the results about asymptotic behavior are obtained under the condition which provides certain proximity of the coefficients $a_T(t,x)$ of equation (1) to some measurable functions $\hat{a}_T(x)$. In such situation, the limit processes, obtained under condition $T \to \infty$, are some functionals of the limits of the solutions $\hat{\xi}_T(t)$ to the homogeneous stochastic differential equations

$$d\hat{\xi}_T(t) = \hat{a}_T\big(\hat{\xi}_T(t)\big)\,dt + dW_T(t). \tag{2}$$

The present paper generalizes similar results from [8] for the unique strong solutions $\hat{\xi}_T$ to homogeneous stochastic differential equations (2) to the case for the solutions $\xi_T(t)$ to inhomogeneous equations (1). Some results about solutions $\hat{\xi}_T$ to homogeneous equations (2), which obtained in [8], have been extended to solutions ξ_T to inhomogeneous equations (1). Under the certain proposed conditions, which present a novelty in comparison with [8], we prove that the asymptotic behavior of the solutions and some functionals of the solutions to inhomogeneous Itô stochastic differential equations (1) is the same as that for the solutions to homogeneous Itô stochastic differential equations (2). The present paper also complements results of paper [9]. Moreover, we assume that the drift coefficient $a_T(t,x)$ in equation (1) can have nonregular dependence on the parameter T. For example, the drift coefficient $a_T(t,x)$ can tend, as $T \to \infty$, to infinity at some points x_k and at some points t_k as well, or it can have degeneracies of some other types. Such a nonregular dependence on T of the coefficients in equation (1) appeared for the first time in [4] and [5], where the limit behavior of the normalized unstable solution of Itô stochastic differential equation, as $t \to \infty$, was investigated for homogeneous equations. In those papers, a special dependence of the coefficients $a_T(x) = \sqrt{T}a(x\sqrt{T})$ on the parameter T was considered with $a(x) \in L_1(\mathbb{R})$. The special dependence of the coefficients $a_T(t,x) = \sqrt{T}a(tT, x\sqrt{T})$ on the parameter T was considered in [6] for inhomogeneous stochastic differential equations (1).

A more detailed review of the known results in this area is presented, for example, in [7] and [8].

The paper is organized as follows. In Section 2, we set the notations and formulate basic definitions. Section 3 contains the statements of the main results. In Section 4, they are proved. Auxiliary results are collected in Section 5. Section 6 gives examples.

2 Main definitions

In what follows we denote by C, L, N, C_N, L_N any constants that do not depend on T, x and t. To formulate and prove the main results we introduce the functions of the form

$$f_T(x) = \int_0^x \exp\left\{-2\int_0^u \hat{a}_T(v)\,dv\right\} du, \quad T > 0. \tag{3}$$

Throughout the paper we use the following notations:

$$\beta_T^{(1)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\,ds, \qquad \beta_T^{(2)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\,dW_T(s),$$

$$I_T(t) = F_T\big(\xi_T(t)\big) + \int_0^t g_T\big(\xi_T(s)\big)\,dW_T(s),$$

where ξ_T and W_T are related via equation (1), $g_T(x)$ is a family of measurable, locally bounded real-valued functions, $F_T(x)$ is a family of continuous real-valued functions.

Definition 2.1. We say that a family of stochastic processes $\zeta_T = \{\zeta_T(t), t \geq 0\}$ weakly converges, as $T \to \infty$, to a process $\zeta = \{\zeta(t), t \geq 0\}$ if, for any $L > 0$, the measures $\mu_T[0, L]$ generated by the processes ζ_T on the interval $[0, L]$ weakly converge to the measure $\mu[0, L]$ generated by the process ζ considered on the interval $[0, L]$.

To study the weak convergence, as $T \to \infty$, of the processes $I_T(t) = F_T(\xi_T(t)) + \int_0^t g_T(\xi_T(s))\,dW_T(s)$, where ξ_T is the solution to stochastic differential equation (1), we suppose additionally that the drift coefficients satisfy the following assumption: there exists a family of measurable, locally bounded functions $\hat{a}_T(x)$ such that for any $L > 0$

$$\lim_{T\to\infty} \int_0^L \sup_x \big|a_T(t, x) - \hat{a}_T(x)\big|\,dt = 0. \tag{A_0}$$

Note, that due to condition (A_0), some results about solutions $\hat{\xi}_T$ to homogeneous equations (2), which are obtained in [8], have been extended to solutions ξ_T to inhomogeneous equations (1). Therefore, by analogy to the paper [8], consider equations (1) from the class $K(G_T)$.

Definition 2.2. The class of equations of the form (1) will be denoted by $K(G_T)$, if there exist families of functions $\hat{a}_T(x)$ and $G_T(x)$, $x \in \mathbb{R}$, such that:

1) $\hat{a}_T(x)$ are measurable locally bounded real-valued functions, satisfying condition (A_0);

2) $G_T(x)$ have continuous derivatives $G'_T(x)$ and locally integrable second derivatives $G''_T(x)$ a.e. with respect to the Lebesgue measure such that, for all $T > 0$, $x \in \mathbb{R}$ and $t \geq 0$, for some constant $C > 0$ the following inequalities hold:

$$\left[G'_T(x)a_T(t, x) + \frac{1}{2}G''_T(x)\right]^2 + \big[G'_T(x)\big]^2 \leq C\big[1 + \big|G_T(x)\big|^2\big],$$

$$\big|G_T(x_0)\big| \leq C; \tag{A_1}$$

3) there exist constants $C > 0$ and $\alpha > 0$ such that, for all $x \in \mathbb{R}$,

$$\left|G_T(x)\right| \geq C|x|^\alpha;$$

4) there exist a bounded function $\psi(|x|)$ and a constant $m \geq 0$ such that $\psi(|x|) \to 0$ as $|x| \to 0$ and, for all $x \in \mathbb{R}$ and $T > 0$ and for any measurable bounded set B, the following inequality holds:

$$\left| f_T'(x) \int_0^x \frac{\chi_B(G_T(u))}{f_T'(u)}\, du \right| \leq \psi\bigl(\lambda(B)\bigr)\bigl[1 + |x|^m\bigr], \tag{A_2}$$

where $\chi_B(x)$ is the indicator function of a set B, $\lambda(B)$ is the Lebesgue measure of B, $f_T'(x)$ is the derivative of the function $f_T(x)$ defined by equality (3).

Assume that, for certain locally bounded functions $q_T(x)$ and any constant $N > 0$, the following condition holds:

$$\lim_{T\to\infty} \sup_{|x|\leq N} \left| f_T'(x) \int_0^x \frac{q_T(v)}{f_T'(v)}\, dv \right| = 0. \tag{A_3}$$

3 Statement of the main results

We are in position to obtain the main result (Theorem 3.1) concerning the weak compactness of stochastic processes $\zeta_T = \{\zeta_T(t) = G_T(\xi_T(t)), t \geq 0\}$, and use it in further investigation of asymptotic behavior of the solutions (Theorem 3.2) and some functionals of the solutions (Theorems 3.3–3.7) to inhomogeneous Itô stochastic differential equations (1).

Theorem 3.1. *Let ξ_T be a solution to equation (1) and let there exist a family of continuous functions $G_T(x)$, $x \in \mathbb{R}$ for which the derivative $G_T'(x)$ is continuous and the second derivative $G_T''(x)$ exists a.e. with respect to the Lebesgue measure and is locally integrable. Let the functions $G_T(x)$ satisfy assumption (A_1), for all $T > 0$, $t \geq 0$, $x \in \mathbb{R}$. Then the family of the processes $\zeta_T = \{\zeta_T(t) = G_T(\xi_T(t)), t \geq 0\}$ is weakly compact.*

Theorem 3.2. *Let ξ_T be a solution to equation (1) from the class $K(G_T)$ and $G_T(x_0) \to y_0$, as $T \to \infty$. Assume that there exist measurable locally bounded functions $a_0(x)$ and $\sigma_0(x)$ such that:*

1) the functions

$$q_T^{(1)}(x) = G_T'(x)\hat{a}_T(x) + \frac{1}{2}G_T''(x) - a_0\bigl(G_T(x)\bigr),$$
$$q_T^{(2)}(x) = \bigl[G_T'(x)\bigr]^2 - \sigma_0^2\bigl(G_T(x)\bigr),$$

satisfy assumption (A_3);

2) the Itô equation

$$\zeta(t) = y_0 + \int_0^t a_0\big(\zeta(s)\big)\,ds + \int_0^t \sigma_0\big(\zeta(s)\big)\,d\hat{W}(s) \tag{4}$$

has a unique weak solution $(\zeta, \hat{W})$.

Then the stochastic processes $\zeta_T = G_T(\xi_T(t))$ *weakly converge, as* $T \to \infty$, *to the solution* ζ *to equation (4).*

Theorem 3.3. *Let* ξ_T *be a solution to equation (1) from the class* $K(G_T)$ *and let assumptions of Theorem 3.2 hold. Assume that for measurable and locally bounded functions* $g_T(x)$ *there exists measurable and locally bounded function* $g_0(x)$ *such that the function*

$$q_T(x) = g_T(x) - g_0\big(G_T(x)\big)$$

satisfies assumption (A_3). *Then the stochastic processes* $\beta_T^{(1)}(t) = \int_0^t g_T(\xi_T(s))\,ds$ *weakly converge, as* $T \to \infty$, *to the process*

$$\beta^{(1)}(t) = \int_0^t g_0\big(\zeta(s)\big)\,ds,$$

where ζ *is a solution to equation (4).*

Theorem 3.4. *Let* ξ_T *be a solution to equation (1) from the class* $K(G_T)$, *and let the assumptions of Theorem 3.2 hold. Assume that, for measurable and locally bounded functions* $g_T(x)$, *there exists a measurable locally bounded function* $g_0(x)$ *such that*

$$\left| f_T'(x) \int_0^x \frac{g_T(v)}{f_T'(v)}\,dv \right| \chi_{|x|\le N} \le C_N,$$

$$\lim_{T\to\infty} \sup_{|x|\le N} \left| f_T'(x) \int_0^x \frac{g_T(v)}{f_T'(v)}\,dv - g_0\big(G_T(x)\big) G_T'(x) \right| = 0 \tag{A_4}$$

for all $N > 0$. *Then the stochastic processes* $\beta_T^{(1)}(t) = \int_0^t g_T(\xi_T(s))\,ds$ *weakly converge, as* $T \to \infty$, *to the process*

$$\tilde{\beta}^{(1)}(t) = 2\left(\int_{y_0}^{\zeta(t)} g_0(x)\,dx - \int_0^t g_0\big(\zeta(s)\big)\sigma_0\big(\zeta(s)\big)\,d\hat{W}(s) \right),$$

where $(\zeta, \hat{W})$ *is a solution to equation (4).*

Theorem 3.5. *Let* ξ_T *be a solution to equation (1) from the class* $K(G_T)$, *and let the assumptions of Theorem 3.2 hold. Suppose that the functions* $\hat{a}_T(x)$ *satisfy assumption* (A_3). *Assume that, for measurable and locally bounded functions* $g_T(x)$, *there exist two constants* c_0 *and* b_0 *such that for all* $N > 0$

$$\left| f_T'(x) \int_0^x \frac{g_T(v)}{f_T'(v)}\,dv \right| \chi_{|x|\le N} \le C_N,$$

$$\lim_{T\to\infty} \sup_{|x|\le N} \left| \int_0^x \left[f_T'(u) \int_0^u \frac{g_T(v)}{f_T'(v)}\,dv - c_0 \right] du \right| = 0,$$

and the functions

$$Q_T(x) = \left[f_T'(x) \int_0^x \frac{g_T(v)}{f_T'(v)}\, dv - c_0 \right]^2 - b_0^2$$

satisfy assumption (A_3). Then the stochastic processes

$$\beta_T^{(1)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\, ds$$

weakly converge, as $T \to \infty$, to the process $2b_0 W(t)$, where $\{W(t), t \geq 0\}$ is a Wiener process.

Theorem 3.6. *Let ξ_T be a solution to equation (1) from the class $K(G_T)$ and let assumptions of Theorem 3.2 hold. Assume that, for measurable and locally bounded functions $g_T(x)$, there exists a measurable locally bounded function $g_0(x)$ such that the function*

$$q_T(x) = \big[g_T(x) - g_0\big(G_T(x)\big)G_T'(x)\big]^2$$

satisfies assumption (A_3). Then the stochastic processes

$$\beta_T^{(2)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\, dW_T(s),$$

where ξ_T and W_T are related via equation (1), weakly converge, as $T \to \infty$, to the process

$$\beta^{(2)}(t) = \int_0^t g_0\big(\zeta(s)\big)\sigma_0\big(\zeta(s)\big)\, d\hat{W}(s),$$

where $(\zeta, \hat{W})$ is a solution to equation (4).

Theorem 3.7. *Let ξ_T and W_T be related via equation (1) from the class $K(G_T)$ and let the assumptions of Theorem 3.2 hold. Assume that, for continuous functions $F_T(x)$ and locally bounded measurable functions $g_T(x)$, there exist a continuous function $F_0(x)$ and locally bounded measurable function $g_0(x)$ such that, for all $N > 0$*

$$\lim_{T\to\infty} \sup_{|x|\leq N} \big|F_T(x) - F_0\big(G_T(x)\big)\big| = 0,$$

and let the functions $g_T(x)$ and $g_0(x)$ satisfy the assumptions of Theorem 3.6. Then the stochastic processes

$$I_T(t) = F_T\big(\xi_T(t)\big) + \int_0^t g_T\big(\xi_T(s)\big)\, dW_T(s)$$

weakly converge, as $T \to \infty$, to the process

$$I_0(t) = F_0\big(\zeta(t)\big) + \int_0^t g_0\big(\zeta(s)\big)\sigma_0\big(\zeta(s)\big)\, d\hat{W}(s),$$

where $(\zeta, \hat{W})$ is a solution to equation (4).

4 Proof of the main results

In the proofs of theorems, which are performed similarly to the proofs of the corresponding theorems in [8], we emphasize the differences associated with inhomogeneous equations. The proof of the Theorem 3.2 is given for a better understanding of the brief proofs of the other theorems.

Proof of Theorem 3.1. The functions $G_T(x)$ have continuous derivatives $G'_T(x)$ for all $T > 0$, their second derivatives $G''_T(x)$ exist a.e. with respect to the Lebesgue measure and are locally integrable. Therefore (see [3], Chap. II, §10), we can apply the Itô formula to the process $\zeta_T(t) = G_T(\xi_T(t))$, and with probability one, for all $t \geq 0$, we obtain

$$\zeta_T(t) = G_T(x_0) + \int_0^t L_T\big(\xi_T(s)\big)\,ds + \int_0^t G'_T\big(\xi_T(s)\big)\,dW_T(s), \tag{5}$$

where

$$L_T(x) = G'_T(x)a_T(t,x) + \frac{1}{2}G''_T(x).$$

Let

$$\chi_N(t) = \begin{cases} 1, & \sup_{0\leq s\leq t}|\zeta_T(s)| \leq N,\\ 0, & \sup_{0\leq s\leq t}|\zeta_T(s)| > N.\end{cases}$$

It is clear that for $s \leq t$ we have $\chi_N(t)\chi_N(s) = \chi_N(t)$ with probability one. Thus, according to (5), the following equality holds with probability one:

$$\begin{aligned}\zeta_T(t)\chi_N(t) = \zeta_T(0)\chi_N(t) + \chi_N(t)\int_0^t L_T\big(\xi_T(s)\big)\chi_N(s)\,ds \\ + \chi_N(t)\int_0^t G'_T\big(\xi_T(s)\big)\chi_N(s)\,dW_T(s).\end{aligned} \tag{6}$$

Hence, using condition (A_1) and the properties of stochastic integrals, we obtain that

$$\begin{aligned}\mathsf{E}\,\zeta_T^2(t)\chi_N(t) &\leq 3\bigg[\mathsf{E}\,\zeta_T^2(0)\chi_N(t) + \mathsf{E}\bigg(\int_0^t L_T\big(\xi_T(s)\big)\chi_N(s)\,ds\bigg)^2 \\ &\quad + \mathsf{E}\bigg(\int_0^t G'_T\big(\xi_T(s)\big)\chi_N(s)\,dW_T(s)\bigg)^2\bigg] \\ &\leq 3\bigg[\mathsf{E}\,\zeta_T^2(0)\chi_N(t) + t\int_0^t \mathsf{E}\,L_T^2\big(\xi_T(s)\big)\chi_N(s)\,ds \\ &\quad + \int_0^t \mathsf{E}\big[G'_T\big(\xi_T(s)\big)\big]^2\chi_N(s)\,ds\bigg] \\ &\leq 3\bigg[C + t\int_0^t C\big[1+\mathsf{E}\,\zeta_T^2(s)\chi_N(s)\big]\,ds \\ &\quad + C\int_0^t \big[1+\mathsf{E}\,\zeta_T^2(s)\chi_N(s)\big]\,ds\bigg] \\ &\leq C_L^{(1)} + C_L^{(2)}\int_0^t \mathsf{E}\,\zeta_T^2(s)\chi_N(s)\,ds,\end{aligned} \tag{7}$$

where $C_L^{(1)} = 3C(1 + t + t^2)$, $C_L^{(2)} = 3C(1 + t)$, $C > 0$ is a constant from condition (A_1), $0 \le t \le L$.

Using the Gronwall–Bellman inequality, we conclude that there exists a constant K_L, which is independent of T, and for $0 \le t \le L$

$$\mathsf{E}\,\zeta_T^2(t)\chi_N(t) \le K_L.$$

Let $N \uparrow \infty$, then $\zeta_T^2(t)\chi_N(t) \uparrow \zeta_T^2(t)$, and we get the inequality

$$\sup_{0\le t\le L} \mathsf{E}\,\zeta_T^2(t) \le K_L. \tag{8}$$

Similarly to (7), using (5) and the inequality

$$\mathsf{E}\sup_{0\le t\le L}\left[\int_0^t G_T'\big(\xi_T(s)\big)\,dW_T(s)\right]^2 \le 4\int_0^L \mathsf{E}\big[G_T'\big(\xi_T(s)\big)\big]^2\,ds,$$

we conclude that

$$\begin{aligned}\mathsf{E}\sup_{0\le t\le L}\big|\zeta_T(t)\big|^2 &\le 3\left[G_T^2(x_0) + L\int_0^L C\big[1+\mathsf{E}\,\zeta_T^2(s)\big]\,ds + \int_0^L C\big[1+\mathsf{E}\,\zeta_T^2(s)\big]\,ds\right]\\ &\le \tilde{C}_L^{(1)} + \tilde{C}_L^{(2)}\int_0^L \mathsf{E}\big[\zeta_T(s)\big]^2\,ds.\end{aligned}$$

Therefore, considering (8), we obtain the inequality

$$\mathsf{E}\sup_{0\le t\le L}\big|\zeta_T(t)\big|^2 \le \tilde{K}_L \tag{9}$$

for all $L > 0$, where the constants $\tilde{K}_L$ are independent of T.

Using the inequalities for martingales and for stochastic integrals (see [2], Part I, §3, Theorem 6), we obtain that

$$\begin{aligned}&\mathsf{E}\sup_{0\le t\le L}\left|\int_0^t G_T'\big(\xi_T(s)\big)\chi_N(s)\,dW_T(s)\right|^{2m}\\ &\quad\le\left(\frac{2m}{2m-1}\right)^{2m}\mathsf{E}\left|\int_0^L G_T'\big(\xi_T(s)\big)\chi_N(s)\,dW_T(s)\right|^{2m}\\ &\quad\le\left(\frac{2m}{2m-1}\right)^{2m}\big[m(2m-1)\big]^{m-1}L^{m-1}\int_0^L \mathsf{E}\big[G_T'\big(\xi_T(s)\big)\big]^{2m}\chi_N(s)\,ds,\end{aligned}$$

for any natural number m. Therefore, similarly to (9) we have inequality

$$\mathsf{E}\sup_{0\le t\le L}\big|\zeta_T(t)\big|^{2m} \le K_{Lm}. \tag{10}$$

Furthermore, for all $\alpha > 0$ there exists $m \in \mathbb{N}$ such that $\alpha \le 2m$ and, for random variable η, we have

$$\mathsf{E}\,|\eta|^\alpha \le 1 + \mathsf{E}\,|\eta|^{2m}.$$

The last inequality together with (10) implies that

$$\mathsf{E}\sup_{0\le t\le L}\left|\zeta_T(t)\right|^{\alpha}\le K_{L\alpha} \tag{11}$$

for all $\alpha>0$ and $L>0$, where the constants $K_{L\alpha}$ are independent of T.

Since for $t_1<t_2\le L$

$$\begin{aligned}
&\mathsf{E}\big[\zeta_T(t_2)-\zeta_T(t_1)\big]^4\\
&\quad\le 8\left[\mathsf{E}\bigg(\int_{t_1}^{t_2}L_T\big(\xi_T(s)\big)\,ds\bigg)^4+\mathsf{E}\bigg(\int_{t_1}^{t_2}G'_T\big(\xi_T(s)\big)\,dW_T(s)\bigg)^4\right]\\
&\quad\le 8\left[(t_2-t_1)^3\int_{t_1}^{t_2}\mathsf{E}\big|L_T\big(\xi_T(s)\big)\big|^4\,ds+36(t_2-t_1)\int_{t_1}^{t_2}\mathsf{E}\big[G'_T\big(\xi_T(s)\big)\big]^4\,ds\right],
\end{aligned}$$

considering condition (A_1) and inequality (11), we get

$$\mathsf{E}\big[\zeta_T(t_2)-\zeta_T(t_1)\big]^4\le C_L|t_2-t_1|^2, \tag{12}$$

where the constants C_L are independent of T.

According to (8) and (12), we have convergences

$$\begin{aligned}
&\lim_{N\to\infty}\overline{\lim_{T\to\infty}}\sup_{0\le t\le L}\mathsf{P}\big\{\left|\zeta_T(t)\right|>N\big\}=0,\\
&\lim_{h\to 0}\overline{\lim_{T\to\infty}}\sup_{\substack{|t_1-t_2|\le h\\ t_i\le L}}\mathsf{P}\big\{\left|\zeta_T(t_2)-\zeta_T(t_1)\right|>\varepsilon\big\}=0
\end{aligned}\tag{13}$$

for any $L>0$, $\varepsilon>0$.

It means that we can apply Skorokhod's convergent subsequence principle (see [11], Chapter I, §6) for the processes $\zeta_T(t)$ for all $0\le t\le L$. According to this principle, given an arbitrary sequence $T'_n\to\infty$, we can choose a subsequence $T_n\to\infty$, a probability space $(\tilde{\Omega},\tilde{\Im},\tilde{\mathsf{P}})$, and stochastic processes $\tilde{\zeta}_{T_n}(t)$, $\zeta(t)$ defined on this space such that their finite-dimensional distributions coincide with those of the processes $\zeta_{T_n}(t)$, and, moreover, $\tilde{\zeta}_{T_n}(t)\xrightarrow{\tilde{\mathsf{P}}}\zeta(t)$, as $T_n\to\infty$, for all $0\le t\le L$. The processes $\tilde{\zeta}_{T_n}(t)$ and $\zeta(t)$ can be considered separable.

Using (12), we have

$$\mathsf{E}\big[\tilde{\zeta}_{T_n}(t_2)-\tilde{\zeta}_{T_n}(t_1)\big]^4\le C_L|t_2-t_1|^2$$

for all $0\le t_1\le t_2\le L$.

By Fatou's lemma,

$$\mathsf{E}\big[\zeta(t_2)-\zeta(t_1)\big]^4\le C_L|t_2-t_1|^2.$$

Thus, the processes $\tilde{\zeta}_{T_n}(t)$ and $\zeta(t)$ are continuous with probability one. We have that finite-dimensional distributions of the processes $\tilde{\zeta}_{T_n}(t)$ converge, as $T_n\to\infty$, to the correspondent finite-dimensional distributions of the process $\zeta(t)$. For a weak convergence of the processes $\zeta_{T_n}(t)$ it is sufficient (see [1], Chapter IX, §2) to prove

$$\lim_{h\to 0}\overline{\lim_{T_n\to\infty}}\mathsf{P}\Big\{\sup_{\substack{|t_1-t_2|\le h\\ t_i\le L}}\big|\zeta_{T_n}(t_2)-\zeta_{T_n}(t_1)\big|>\varepsilon\Big\}=0 \tag{14}$$

for any $L>0$, $\varepsilon>0$.

Due to inequalities (see [1], Chapter IX, §3)

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{\substack{|t_1-t_2|\le h\\ t_i\le L}}\big|\zeta_{T_n}(t_2)-\zeta_{T_n}(t_1)\big|>\varepsilon\Big\}\\
&\le\sum_{kh\le L}\mathsf{P}\Big\{\sup_{kh\le t\le(k+1)h}\big|\zeta_{T_n}(t)-\zeta_{T_n}(kh)\big|>\frac{\varepsilon}{4}\Big\}\\
&\le\Big(\frac{4}{\varepsilon}\Big)^4 8\sum_{kh\le L}\bigg\{\mathsf{E}\sup_{kh\le t\le(k+1)h}\bigg(\int_{kh}^t L_{T_n}\big(\xi_T(s)\big)\,ds\bigg)^4\\
&\quad+\mathsf{E}\sup_{kh\le t\le(k+1)h}\bigg(\int_{kh}^t G'_{T_n}\big(\xi_T(s)\big)\,dW_T(s)\bigg)^4\bigg\}\le\Big(\frac{4}{\varepsilon}\Big)^4 K_L\sum_{kh\le L}\big(h^4+h^2\big),
\end{aligned}$$

where K_L are independent of T_n, we obtain (14). The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2. Rewrite equality (5) as

$$\zeta_T(t)=G_T(x_0)+\int_0^t a_0\big(\zeta_T(s)\big)\,ds+\eta_T(t)+\alpha_T^{(0)}(t)+\alpha_T^{(1)}(t), \tag{15}$$

where

$$\begin{aligned}
\eta_T(t)&=\int_0^t G'_T\big(\xi_T(s)\big)\,dW_T(s),\\
\alpha_T^{(0)}(t)&=\int_0^t G'_T\big(\xi_T(s)\big)\Delta a_T(s)\,ds,\qquad \Delta a_T(s)=a_T\big(s,\xi_T(s)\big)-\hat{a}_T\big(\xi_T(s)\big),\\
\alpha_T^{(1)}(t)&=\int_0^t q_T^{(1)}\big(\xi_T(s)\big)\,ds,\qquad q_T^{(1)}(x)=G'_T(x)\hat{a}_T(x)+\frac{1}{2}G''_T(x)-a_0\big(G_T(x)\big).
\end{aligned}$$

The conditions (A_0) and (A_1), together with inequality (11), imply that

$$\begin{aligned}
\sup_{0\le t\le L}\big|\alpha_T^{(0)}(t)\big|&\le\int_0^L\big|G'_T\big(\xi_T(s)\big)\big|\big|\Delta a_T(s)\big|\,ds\\
&\le\Big[C\Big(1+\sup_{0\le s\le L}\big|\zeta_T(s)\big|^2\Big)\Big]^{\frac{1}{2}}\int_0^L\sup_x\big|a_T(s,x)-\hat{a}_T(x)\big|\,ds\xrightarrow{\mathsf{P}}0,
\end{aligned} \tag{16}$$

as $T\to\infty$ for any $L>0$.

The functions $q_T^{(1)}(x)$ satisfy conditions of Lemma 5.2. Thus, for any $L > 0$

$$\sup_{0\le t\le L} \left|\alpha_T^{(1)}(t)\right| \xrightarrow{\mathsf{P}} 0, \tag{17}$$

as $T \to \infty$.

It is clear that $\eta_T(t)$ is a family of continuous martingales with quadratic characteristics

$$\langle \eta_T\rangle(t) = \int_0^t \left[G_T'\left(\xi_T(s)\right)\right]^2 ds = \int_0^t \sigma_0^2\left(\zeta_T(s)\right) ds + \alpha_T^{(2)}(t), \tag{18}$$

where

$$\alpha_T^{(2)}(t) = \int_0^t q_T^{(2)}\left(\xi_T(s)\right) ds, \quad q_T^{(2)}(x) = \left(G_T'(x)\right)^2 - \sigma_0^2\left(G_T(x)\right).$$

The functions $q_T^{(2)}(x)$ satisfy conditions of Lemma 5.2. Thus, for any $L > 0$

$$\sup_{0\le t\le L} \left|\alpha_T^{(2)}(t)\right| \xrightarrow{\mathsf{P}} 0, \tag{19}$$

as $T \to \infty$.

According to Theorem 3.1, the family of the processes $\zeta_T(t)$ is weakly compact. It is easy to see that compactness conditions (14) are fulfilled for the processes $\eta_T(t)$. Using convergences (16), (17), (19), we have that relations (14) hold for the processes $\alpha_T^{(k)}(t)$, $k = 0, 1, 2$ as well. It means that we can apply Skorokhod's convergent subsequence principle (see [11], Chapter I, §6) for the processes

$$\left(\zeta_T(t),\ \eta_T(t),\ \alpha_T^{(k)}(t), \quad k = 0, 1, 2\right).$$

According to this principle, given an arbitrary sequence $T_n' \to \infty$, we can choose a subsequence $T_n \to \infty$, a probability space $(\tilde{\Omega}, \tilde{\Im}, \tilde{\mathsf{P}})$, and stochastic processes

$$\left(\tilde{\zeta}_{T_n}(t),\ \tilde{\eta}_{T_n}(t),\ \tilde{\alpha}_{T_n}^{(k)}(t), \quad k = 0, 1, 2\right)$$

defined on this space such that their finite-dimensional distributions coincide with those of the processes

$$\left(\zeta_{T_n}(t),\ \eta_{T_n}(t),\ \alpha_{T_n}^{(k)}(t), \quad k = 0, 1, 2\right),$$

and, moreover,

$$\tilde{\zeta}_{T_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde{\zeta}(t), \qquad \tilde{\eta}_{T_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde{\eta}(t), \qquad \tilde{\alpha}_{T_n}^{(k)}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde{\alpha}^{(k)}(t), \quad k = 0, 1, 2,$$

as $T_n \to \infty$, for all $0 \le t \le L$, where $\tilde{\zeta}(t)$, $\tilde{\eta}(t)$, $\tilde{\alpha}^{(k)}(t), k = 0, 1, 2$ are some stochastic processes.

Evidently, relations (16)–(19) imply that $\tilde{\alpha}^{(k)}(t) \equiv 0$, $k = 0, 1, 2$ a.s. According to (12), the processes $\tilde{\zeta}(t)$ and $\tilde{\eta}(t)$ are continuous with probability one. Moreover, applying Lemma 5.3 together with equalities (15) and (18), we obtain that

$$\tilde{\zeta}_{T_n}(t) = G_{T_n}(x_0) + \int_0^t a_0\big(\tilde{\zeta}_{T_n}(s)\big)\,ds + \tilde{\alpha}^{(0)}_{T_n}(t) + \tilde{\alpha}^{(1)}_{T_n}(t) + \tilde{\eta}_{T_n}(t), \tag{20}$$

$$\langle\tilde{\eta}_{T_n}\rangle(t) = \int_0^t \sigma_0^2\big(\tilde{\zeta}_{T_n}(s)\big)\,ds + \tilde{\alpha}^{(2)}_{T_n}(t),$$

where

$$\tilde{\zeta}_{T_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde{\zeta}(t), \qquad \tilde{\eta}_{T_n}(t) \xrightarrow{\tilde{\mathsf{P}}} \tilde{\eta}(t), \qquad \sup_{0\le t\le L}\big|\tilde{\alpha}^{(k)}_{T_n}(t)\big| \xrightarrow{\tilde{\mathsf{P}}} 0, \quad k = 0, 1, 2,$$

as $T_n \to \infty$.

An analogue of convergence (14) holds for the processes $\tilde{\zeta}_{T_n}(t)$ and $\tilde{\eta}_{T_n}(t)$. Therefore, according to the well-known result of Prokhorov (Lemma 1.11 in [10]), we conclude that for any $L > 0$

$$\sup_{0\le t\le L}\big|\tilde{\zeta}_{T_n}(t) - \tilde{\zeta}(t)\big| \xrightarrow{\tilde{\mathsf{P}}} 0, \qquad \sup_{0\le t\le L}\big|\tilde{\eta}_{T_n}(t) - \tilde{\eta}(t)\big| \xrightarrow{\tilde{\mathsf{P}}} 0, \tag{21}$$

as $T_n \to \infty$.

According to Lemma 5.4, we can pass to the limit in (20) and obtain

$$\tilde{\zeta}(t) = y_0 + \int_0^t a_0\big(\tilde{\zeta}(s)\big)\,ds + \tilde{\eta}(t),$$

where $\tilde{\eta}(t)$ is the almost surely continuous martingale with the quadratic characteristic

$$\langle\tilde{\eta}\rangle(t) = \int_0^t \sigma_0^2\big(\tilde{\zeta}(s)\big)\,ds.$$

Now, it is well known, that the latter representation provides the existence of a Wiener process $\hat{W}(t)$ such that

$$\tilde{\eta}(t) = \int_0^t \sigma_0\big(\tilde{\zeta}(s)\big)\,d\hat{W}(s).$$

Thus, the process $(\tilde{\zeta}, \hat{W})$ satisfies equation (4), and the processes $\tilde{\zeta}_{T_n}(t)$ weakly converge, as $T_n \to \infty$, to the process $\tilde{\zeta}$. Since the sequence $T_n^{'} \to \infty$ is arbitrary and since a solution to equation (4) is weakly unique, the proof of the Theorem 3.2 is complete. □

The proof of Theorems 3.3–3.4, 3.6–3.7 is performed similarly to the proof of the corresponding theorems in [8] with some differences that we discuss below.

Remark 4.1. The proof of Theorem 3.4 differs from the proof of Theorem 2.3 in [8] by using other representation for the functional $\beta_T^{(1)}(t) = \int_0^t g_T(\xi_T(s))\,ds$. In this case we have

$$\beta_T^{(1)}(t) = 2\int_{G_T(x_0)}^{\zeta_T(t)} g_0(u)\,du - 2\int_0^t g_0\big(\zeta_T(s)\big)\,d\eta_T(s) + \gamma_T^{(1)}(t) - \gamma_T^{(2)}(t) - \gamma_T^{(0)}(t),$$

where

$$\gamma_T^{(1)}(t)=\int_{x_0}^{\xi_T(t)}\hat q_T(u)\,du,\qquad \gamma_T^{(2)}(t)=\int_0^t \hat q_T\big(\xi_T(s)\big)\,dW_T(s),$$

$$\gamma_T^{(0)}(t)=\int_0^t \Phi_T'\big(\xi_T(s)\big)\big[a_T\big(s,\xi_T(s)\big)-\hat a_T\big(\xi_T(s)\big)\big]\,ds,$$

$$\Phi_T(x)=2\int_0^x f_T'(u)\bigg(\int_0^u \frac{g_T(v)}{f_T'(v)}\,dv\bigg)\,du,$$

$$\hat q_T(x)=\Phi_T'(x)-2g_0\big(G_T(x)\big)G_T'(x),$$

$f_T'(x)$ is the derivative of the function $f_T(x)$ defined by equality (3).

The latter representation differs from the corresponding representation in [8] by the term $\gamma_T^{(0)}(t)$. For any constants $\varepsilon>0$, $N>0$ and $L>0$, we have the inequalities

$$\begin{aligned}\mathsf{P}\Big\{\sup_{0\le t\le L}\big|\gamma_T^{(0)}(t)\big|>\varepsilon\Big\}&\le P_{NT}+\frac{2}{\varepsilon}\int_0^L \mathsf{E}\big|\Phi_T'\big(\xi_T(s)\big)\big|\\ &\quad\times\big|a_T\big(s,\xi_T(s)\big)-\hat a_T\big(\xi_T(s)\big)\big|\chi_{|\xi_T(s)|\le N}\,ds\\ &\le P_{NT}+\frac{2}{\varepsilon}C_N\int_0^L \sup_x\big[a_T(s,x)-\hat a_T(x)\big]\,ds,\end{aligned}$$

where $P_{NT}=\mathsf{P}\{\sup_{0\le t\le L}|\xi_T(t)|>N\}$. Using condition 3) from Definition 2.2 and inequality (11), we obtain the convergence $\lim_{N\to\infty}\overline{\lim}_{T\to\infty}P_{NT}=0$. Using the assumptions of Theorem 3.4, we conclude that

$$\sup_{0\le t\le L}\big|\gamma_T^{(0)}(t)\big|\xrightarrow{\mathsf{P}}0 \tag{22}$$

for any $L>0$, as $T\to\infty$. The rest of the proof of Theorem 3.4 is the same as that of Theorem 2.3 in [8].

The proofs of Theorem 3.3 and Theorems 3.6–3.7 are literally the same as those of Theorem 2.2 and Theorems 2.4–2.5 from [8].

Proof of Theorem 3.5. For the functional $\beta_T^{(1)}(t)=\int_0^t g_T(\xi_T(s))\,ds$, with probability one, for all $t\ge 0$, we have the representation

$$\beta_T^{(1)}(t)=2c_0\textstyle\int_0^t \hat a_T(\xi_T(s))\,ds+\gamma_T(t)-\eta_T^{(1)}(t)-\gamma_T^{(0)}(t)+\gamma_T^{(3)}(t),$$

where

$$\gamma_T(t)=2\int_{x_0}^{\xi_T(t)}\bigg[f_T'(u)\int_0^u \frac{g_T(v)}{f_T'(v)}\,dv-c_0\bigg]\,du,$$

$$\eta_T^{(1)}(t)=\int_0^t\big[\Phi_T'\big(\xi_T(s)\big)-2c_0\big]\,dW_T(s),$$

$$\gamma_T^{(3)}(t)=2c_0\int_0^t\big[a_T\big(s,\xi_T(s)\big)-\hat a_T\big(\xi_T(s)\big)\big]\,ds,$$

$\gamma_T^{(0)}(t)$ and $\Phi_T(x)$ are defined in Remark 4.1.

The functions $\hat{a}_T(x)$ satisfy condition (A_3). Thus, using Lemma 5.2, we conclude that for any $L > 0$

$$\sup_{0\le t\le L}\left|\int_0^t \hat{a}_T\big(\xi_T(s)\big)\,ds\right| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$.

For any constants $\varepsilon > 0$, $N > 0$ and $L > 0$, we have the inequalities

$$\mathsf{P}\Big\{\sup_{0\le t\le L}\big|\gamma_T(t)\big| > \varepsilon\Big\} \le P_{NT} + \frac{1}{\varepsilon}\mathsf{E}\sup_{0\le t\le L}\left|\int_{x_0}^{\xi_T(t)}\big[\Phi_T'(u) - 2c_0\big]\,du\right|\chi_{\{|\xi_T(t)|\le N\}}$$
$$\le P_{NT} + \frac{2}{\varepsilon}N\sup_{|x|\le N}\left|\int_{x_0}^{x}\left[f_T'(u)\int_0^u \frac{g_T(v)}{f_T'(v)}\,dv - c_0\right]du\right|,$$

where P_{NT} is the same as that in Remark 4.1. Using the latter inequality and the assumptions of Theorem 3.5, we conclude that

$$\sup_{0\le t\le L}\big|\gamma_T(t)\big| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$.

Since the term $\gamma_T^{(0)}(t)$ is the same as that in Remark 4.1, we have (22).

The inequality

$$\sup_{0\le t\le L}\big|\gamma_T^{(3)}(t)\big| \le 2c_0\int_0^L \sup_x\big[a_T(s,x) - \hat{a}_T(x)\big]\,ds$$

implies that for any $L > 0$

$$\sup_{0\le t\le L}\big|\gamma_T^{(3)}(t)\big| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$.

Thus, we have for any $L > 0$

$$\sup_{0\le t\le L}\big|\beta_T^{(1)}(t) + \eta_T^{(1)}(t)\big| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$.

It is clear that $\eta_T^{(1)}(t)$ is the almost surely continuous martingale with the quadratic characteristic

$$\langle\eta_T^{(1)}\rangle(t) = 4b_0^2 t + \int_0^t q_T\big(\xi_T(s)\big)\,ds,$$

where $q_T(x) = [\Phi_T'(x) - 2c_0]^2 - 4b_0^2$. The functions $q_T(x)$ satisfy condition (A_3). Thus, using Lemma 5.2, we conclude that for any $L > 0$

$$\sup_{0\le t\le L}\big|\langle\eta_T^{(1)}\rangle(t) - 4b_0^2 t\big| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$.

Then, using a random change of time (see [2], Part I, §4), we obtain $\eta_T^{(1)}(t) = W_T^*(\langle\eta_T^{(1)}\rangle(t))$, where $W_T^*(t)$ is a Wiener process. The same arguments as those used to get (9) in [7] yield that

$$\sup_{0\le t\le L}\left|\beta_T^{(1)}(t) - W_T^*\left(4b_0^2 t\right)\right| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$. Thus, the processes $\beta_T^{(1)}(t)$ weakly converge, as $T \to \infty$, to the process $2b_0 W(t)$. □

5 Auxiliary results

Lemma 5.1. *Let ξ_T be a solution to equation (1) from the class $K(G_T)$. Then, for any $N > 0$, $L > 0$ and any Borel set $B \subset [-N; N]$, there exists a constant C_L such that*

$$\int_0^L \mathsf{P}\left\{G_T\left(\xi_T(s)\right) \in B\right\} ds \le C_L\psi\left(\lambda(B)\right),$$

where $\lambda(B)$ is the Lebesgue measure of the set B, $\psi(|x|)$ is a certain bounded function satisfying $\psi(|x|) \to 0$ as $|x| \to 0$.

Proof. Consider the function

$$\Phi_T(x) = 2\int_0^x f_T'(u)\left(\int_0^u \frac{\chi_B(G_T(v))}{f_T'(v)}\,dv\right)du.$$

The function $\Phi_T(x)$ is continuous, the derivative $\Phi_T'(x)$ of this function is continuous and the second derivative $\Phi_T''(x)$ exists a.e. with respect to the Lebesgue measure and is locally integrable. Therefore, we can apply the Itô formula to the process $\Phi_T(\xi_T(t))$, where $\xi_T(t)$ is a solution to equation (1).

Furthermore,

$$\Phi_T'(x)\hat{a}_T(x) + \frac{1}{2}\Phi_T''(x) = \chi_B(x),$$

a.e. with respect to the Lebesgue measure. Using the Itô formula and the latter equality, we conclude that

$$\int_0^L \chi_B\left(\zeta_T(s)\right) ds = \Phi_T\left(\xi_T(L)\right) - \Phi_T(x_0) - \int_0^L \Phi_T'\left(\xi_T(s)\right) dW_T(s) - \alpha_T(L),$$

with probability one for all $t \ge 0$, where $\zeta_T(t) = G_T(\xi_T(t))$,

$$\alpha_T(t) = \int_0^t \Phi_T'\left(\xi_T(s)\right)\left[a_T\left(s, \xi_T(s)\right) - \hat{a}_T\left(\xi_T(s)\right)\right] ds.$$

Hence, using the properties of stochastic integrals, we obtain that

$$\int_0^L \mathsf{P}\left\{\zeta_T(s) \in B\right\} ds = \mathsf{E}\left[\Phi_T\left(\xi_T(L)\right) - \Phi_T(x_0)\right] - \mathsf{E}\,\alpha_T(L). \qquad (23)$$

According to condition (A_2), inequalities $|G_T(x)| \geq C|x|^\alpha$, $C > 0$, $\alpha > 0$ and (11), we have

$$\big|\mathsf{E}\big[\Phi_T\big(\xi_T(L)\big) - \Phi_T(x_0)\big]\big| \leq C_L^{(1)}\psi\big(\lambda(B)\big),$$

for a certain constant $C_L^{(1)}$. Condition (A_0) implies that

$$\big|\mathsf{E}\,\alpha_T(L)\big| \leq C_L^{(2)}\psi\big(\lambda(B)\big),$$

for a certain constant $C_L^{(2)}$. Here the function $\psi(\lambda(B))$ is from condition (A_2). The latter inequalities and equality (23) prove Lemma 5.1. □

Lemma 5.2. *Let ξ_T be a solution to equation (1) from the class $K(G_T)$. If, for measurable locally bounded functions $q_T(x)$, condition (A_3) holds, then, for any $L > 0$,*

$$\sup_{0\leq t\leq L}\left|\int_0^t q_T\big(\xi_T(s)\big)\,ds\right| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$.

Proof. Consider the function

$$\Phi_T(x) = 2\int_0^x f_T'(u)\left(\int_0^u \frac{q_T(v)}{f_T'(v)}\,dv\right)du.$$

The function $\Phi_T(x)$ is continuous, the derivative $\Phi_T'(x)$ of this function is continuous and the second derivative $\Phi_T''(x)$ exists a.e. with respect to the Lebesgue measure and is locally integrable. Therefore, we can apply the Itô formula to the process $\Phi_T(\xi_T(t))$, where $\xi_T(t)$ is a solution to equation (1).

Furthermore,

$$\Phi_T'(x)\hat{a}_T(x) + \frac{1}{2}\Phi_T''(x) = q_T(x),$$

a. e. with respect to the Lebesgue measure. Using the latter equality, we conclude that with probability one for all $t \geq 0$

$$\int_0^t q_T\big(\xi_T(s)\big)\,ds = \Phi_T\big(\xi_T(t)\big) - \Phi_T(x_0) - \int_0^t \Phi_T'\big(\xi_T(s)\big)\,dW_T(s) - \alpha_T(t), \quad (24)$$

where

$$\alpha_T(t) = \int_0^t \Phi_T'\big(\xi_T(s)\big)\big[a_T\big(s,\xi_T(s)\big) - \hat{a}_T\big(\xi_T(s)\big)\big]\,ds.$$

For any constants $\varepsilon > 0$, $N > 0$ and $L > 0$, we have

$$\mathsf{P}\Big\{\sup_{0\leq t\leq L}\big|\alpha_T(t)\big| > \varepsilon\Big\} \leq P_{NT} + \frac{2}{\varepsilon}\sup_{|x|\leq N} f_T'(x)\left|\int_0^x \frac{q_T(v)}{f_T'(v)}\,dv\right| \times \int_0^L \sup_x\big[a_T(s,x) - \hat{a}_T(x)\big]\,ds,$$

where $P_{NT} = \mathsf{P}\{\sup_{0\leq t\leq L}|\xi_T(t)| > N\}$.

The same arguments as those used in [8] (see the proof of Lemma 4.2) and the assumptions of Lemma 5.2 yield that

$$\sup_{0\le t\le L} \left|\alpha_T(t)\right| \xrightarrow{\mathsf{P}} 0,$$

$$\sup_{0\le t\le L} \left|\Phi_T\big(\xi_T(t)\big) - \Phi_T(x_0)\right| \xrightarrow{\mathsf{P}} 0,$$

$$\sup_{0\le t\le L} \left|\int_0^t \Phi_T'\big(\xi_T(s)\big)\, dW_T(s)\right| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$. Thus, equality (24) implies the statement of Lemma 5.2. □

The statements and the proofs of the following lemmas are the same as those of the corresponding lemmas from [8].

Lemma 5.3. *Let ξ_T be a solution to equation (1) belonging to the class $K(G_T)$, and let the stochastic process $(\zeta_T, \ \eta_T)$, with $\zeta_T(t) = G_T(\xi_T(t))$ and $\eta_T(t) = \int_0^t G_T'(\xi_T(s))\, dW_T(s)$ be stochastically equivalent to the process $(\tilde{\zeta}_T, \ \tilde{\eta}_T)$. Then the process*

$$\int_0^t g\big(\zeta_T(s)\big)\, ds + \int_0^t q\big(\zeta_T(s)\big)\, d\eta_T(s),$$

where $g(x)$ and $q(x)$ are measurable locally bounded functions, is stochastically equivalent to the process

$$\int_0^t g\big(\tilde{\zeta}_T(s)\big)\, ds + \int_0^t q\big(\tilde{\zeta}_T(s)\big)\, d\tilde{\eta}_T(s).$$

Lemma 5.4. *Let ξ_T be a solution to equation (1) from the class $K(G_T)$, and let $\zeta_T(t) = G_T(\xi_T(t)) \xrightarrow{\mathsf{P}} \zeta(t)$ as $T \to \infty$. Then for any measurable locally bounded function $g(x)$, we have the convergence*

$$\sup_{0\le t\le L} \left|\int_0^t g\big(\zeta_T(s)\big)\, ds - \int_0^t g\big(\zeta(s)\big)\, ds\right| \xrightarrow{\mathsf{P}} 0,$$

as $T \to \infty$ for any constant $L > 0$.

6 Examples

We denote by b_T the family of such constants that $b_T > 1$ and $b_T \uparrow \infty$ as $T \to \infty$.

Example 6.1. Consider equation (1) with the drift coefficient with nonregular dependence on the parameter T of the form

$$a_T(t,x) = b_T^{\gamma} \cos(x b_T) + \frac{t b_T}{1 + t^2 b_T^2} \sin\big((x-1) b_T\big), \quad 0 \le \gamma < 1.$$

The family of measurable locally bounded real-valued functions $\hat{a}_T(x) = b_T^{\gamma}\cos(xb_T)$ satisfies condition 1) from Definition 2.2: for any $L > 0$

$$\lim_{T\to\infty}\int_0^L \sup_x \big|a_T(t,x) - \hat{a}_T(x)\big|\,dt \le \lim_{T\to\infty}\int_0^L \frac{tb_T}{1+t^2b_T^2}\,dt = 0.$$

The rest of conditions from Definition 2.2 are fulfilled for the family of functions

$$G_T(x) = f_T(x) = \int_0^x \exp\bigg\{-2\int_0^u \hat{a}_T(v)\,dv\bigg\}\,du, \quad T > 0.$$

Since $f_T'(x) = \exp\{-2\frac{b_T^{\gamma}}{b_T}\sin(xb_T)\}$, then there exist two constants c_0 and δ_0 such that, for all $x \in \mathbb{R}$, we have $0 < \delta_0 \le f_T'(x) \le c_0$. Taking into account that $G_T(x) = \int_0^x f_T'(v)\,dv$, we obtain $G_T'(x)\hat{a}_T(x) + \frac{1}{2}G_T''(x) \equiv 0$. Therefore, the conditions of Definition 2.2 are fulfilled as follows:

condition 2)

$$\begin{aligned}
&\bigg[G_T'(x)a_T(t,x) + \frac{1}{2}G_T''(x)\bigg]^2 + \big[G_T'(x)\big]^2\\
&\quad = \bigg[G_T'(x)\frac{tb_T}{1+t^2b_T^2}\sin\big((x-1)b_T\big)\bigg]^2 + \big[G_T'(x)\big]^2 \le 2\big[G_T'(x)\big]^2\\
&\quad \le 2c_0^2 \le 2c_0^2\big[1 + \big|G_T(x)\big|^2\big];\\
&\big|G_T(x_0)\big| = \bigg|\int_0^{x_0} f_T'(v)\,dv\bigg| \le c_0\cdot|x_0| = C;
\end{aligned}$$

condition 3)

$$\big|G_T(x)\big| = \bigg|\int_0^x f_T'(v)\,dv\bigg| \ge C|x|^{\alpha} \quad \text{with } C = \delta_0,\ \alpha = 1;$$

condition 4)

$$\begin{aligned}
&\bigg|\int_0^x f_T'(u)\bigg(\int_0^u \frac{\chi_B(G_T(v))}{f_T'(v)}\,dv\bigg)\,du\bigg|\\
&\quad \le \frac{C_0}{\delta_0}\bigg|\int_0^x\int_0^u \chi_B\big(G_T(v)\big)\,dv\,du\bigg| \le C_1\lambda(B)|x| \le \psi\big(\lambda(B)\big)\big[1+|x|^m\big]
\end{aligned}$$

with $\psi(|x|) = C_1|x|$, $m = 1$.

Thus, equation (1) belongs to the class $K(G_T)$. According to Theorem 3.1, the family of the processes $\zeta_T(t) = G_T(\xi_T(t))$ is weakly compact. We can find the form of the limit process using Theorem 3.2 with $a_0(x) \equiv 0$, $\sigma_0(x) \equiv 1$. According to Theorem 3.2, the stochastic processes $\zeta_T(t)$ weakly converge, as $T \to \infty$, to the solution $\zeta(t)$ to equation (4) and the limit process is $\zeta(t) = x_0 + \hat{W}(t)$, where $\hat{W}(t)$ is a Wiener process.

Example 6.2. Let the conditions of Example 6.1 hold. For the family of functions

$$g_T(x) = \frac{b_T^{\gamma}}{1 + b_T^2 x^2}, \quad 0 \le \gamma < 1,$$

the assumptions of Theorem 3.3 hold with $g_0(x) \equiv 0$.

According to Theorem 3.3, the stochastic processes

$$\beta_T^{(1)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\, ds = \int_0^t \frac{b_T^{\gamma}}{1 + b_T^2 \xi_T^2(s)}\, ds, \quad 0 \le \gamma < 1$$

weakly converge, as $T \to \infty$, to the process $\beta^{(1)}(t) \equiv 0$.

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