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On singularity of distribution of random variables with independent symbols of Oppenheim expansions

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Abstract The paper is devoted to the restricted Oppenheim expansion of real numbers (ROE), which includes already known Engel, Sylvester and Lüroth expansions as partial cases. We find conditions under which for almost all (with respect to Lebesgue measure) real numbers from the unit interval their ROE-expansion contain arbitrary digit i only finitely many times. Main results of the paper state the singularity (w.r.t. the Lebesgue measure) of the distribution of a random variable with i.i.d. increments of symbols of the restricted Oppenheim expansion. General non-i.i.d. case is also studied and sufficient conditions for the singularity of the corresponding probability distributions are found.

Keywords Restricted Oppenheim expansion, singular probability distributions, metric theory of ROE, Sylvester expansion
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1 Introduction

Singularly continuous probability measures were studied during almost all XX century and there are a lot of open problems related to them. The fractal and multifractal approaches to the study of such measures are known to be extremely useful (see, e.g.,

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[7, 12, 39] and references therein). The study of fractal properties of different families of singularly continuous probability measures (see, e.g., [7, 16, 26, 28, 27, 29, 31, 38, 1, 42] and references therein) can be used to solve non-trivial problems in the metric number theory ([8, 9, 5, 4, 10, 30, 21]), in the theory of dynamical systems and DP-transformations and in fractal analysis ([6, 3, 11, 19, 18, 17, 22, 41, 43]).

On the other hand, for many families of probability measures the problem "singularity vs absolute continuity" is extremely complicated even for the so-called probability distributions of the Jessen–Wintner type, i.e., distributions of random variables which are sums of almost surely convergent series of independent discretely distributed random variables (such probability distributions are of pure type [23]). The Lévy theorem [25] gives necessary and sufficient conditions for such measures to be discrete resp. continuous, and the main problem is to find sharp conditions for absolute resp. singular continuity. Infinite Bernoulli convolutions make an important subclass of such measures (see, e.g., [2, 32, 34, 36, 35, 37] and references therein). Another wide family of probability distributions where the problem "singularity vs absolute continuity" is still open consists of probability distributions of the following form:

$$\xi = \Delta^F_{\xi_1\xi_2\ldots\xi_n\ldots},$$

where ξ_n are independent symbols of some generalized F-expansion over some alphabet A. Random variables with independent symbols of s-adic expansions, continued fraction expansions, the Lüroth expansion, the Sylvester and Engel expansions are among them. This paper is devoted to the development of probabilistic theory of Oppenheim expansions of real numbers which contains many important expansions as rather special cases. Let us mention that many authors studied normal properties of real numbers in terms of digits of their Oppenheim expansion and the Hausdorff dimension of corresponding exceptional sets (see, e.g., [13, 15, 44, 45, 24]). In Section 2 we develop approach which has been invented by G. Torbin to study normal properties of the Ostrogradsky–Sierpinski–Pierce expansion [40] and get general results on normal properties of Oppenheim expansions. Based on these results in the last section of the paper we show that singularity is typical for the family of probability measures with independent symbols of ROE expansions.

2 On metric theory of the restricted Oppenheim expansion

It is known ([14]) that any real number $x \in (0, 1)$ can be represented in the form of the Oppenheim expansion

$$x \sim \frac{1}{d_1} + \frac{a_1}{b_1}\frac{1}{d_2} + \cdots + \frac{a_1 a_2 \cdot \ldots \cdot a_n}{b_1 b_2 \cdot \ldots \cdot b_n}\frac{1}{d_{n+1}} + \cdots \tag{1}$$

where $a_n = a_n(d_1, \ldots, d_n)$, $b_n = b_n(d_1, \ldots, d_n)$ are positive integer valued functions and the denominators d_n are determined by the following procedure: for a given x we define the sequences $\{x_n\}$ and $\{d_n\}$ via

$$x_1 := x;$$

$$d_n = \left[\frac{1}{x_n}\right] + 1;$$

$$x_{n+1} := \frac{b_n}{a_n}\left(x_n - \frac{1}{d_n}\right). \tag{2}$$

A sufficient condition for a series on the right-hand side in (1) to be the expansion of its sum is:

$$d_{n+1} \geq \frac{a_n}{b_n} d_n(d_n - 1) + 1.$$

We call the expansion (1) the restricted Oppenheim expansion (ROE) of x if a_n and b_n depend only on the last denominator d_n and if the function

$$h_n(j) := \frac{a_n(j)}{b_n(j)} j(j-1) \tag{3}$$

is integer valued.

Let us consider some examples of the restricted Oppenheim expansions.

Example 1. Let $a_n = 1, b_n = d_n$ $(n = 1, 2, \ldots)$. Then the expansion (1) obtained by the algorithm (2) is the well-known Engel expansion of x:

$$x = \frac{1}{d_1} + \frac{1}{d_1 d_2} + \cdots + \frac{1}{d_1 d_2 \ldots d_n} + \cdots,$$

where $d_{n+1} \geq d_n$.

Example 2. Let $a_n = b_n = 1$ (or $a_n = b_n = const$) $(n = 1, 2, \ldots)$. Then the expansion (1) obtained by the algorithm (2) is the well-known Sylvester expansion of x:

$$x = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} + \cdots,$$

where $d_{n+1} \geq d_n(d_n - 1) + 1$.

Example 3. Let $a_n = 1, b_n = d_n(d_n - 1)$. In this case we obtain the Lüroth series for a number x:

$$x = \frac{1}{d_1} + \frac{1}{d_1(d_1 - 1)d_2} + \cdots + \frac{1}{d_1(d_1 - 1)\ldots d_n(d_n - 1)d_{n+1}} + \cdots,$$

where $d_{n+1} \geq 2$.

Let us mention that metric, dimensional and probabilistic theories of Oppenheim series are not sufficiently developed. In fact, as evidenced by recent works and thesis in the field ([46, 20, 33]), even such partial cases of Oppenheim expansions as the Lüroth series, the Engel and Sylvester series generate a number of challenges for the metric and probabilistic number theory. The main purpose of this article is to develop some general methods of the metric theory of numbers and Oppenheim expansions and to show their effectiveness in the study of Lebesgue structures of distributions of random variables with independent symbols of Oppenheim expansions.

Choose the probability space $(\Omega, \mathcal{A}, P)$, with $\Omega = (0, 1)$, $\mathcal{A}$ the set of Lebesgue measurable subsets of $(0, 1)$ and the Lebesgue measure as P.

Let $\triangle^{ROE}_{j_1 j_2 \ldots j_n} := \{x : d_1(x) = j_1, d_2(x) = j_2, \ldots, d_n(x) = j_n\}$ be the cylinder of rank n with base $(j_1, j_2, \ldots, j_n)$.

Lemma 1. ([14]) Let x be the random variable, which is uniformly distributed on the unit interval and let $d_j := d_j(x)$. Then

$$P(d_1 = j_1, \dots, d_n = j_n) = \frac{a_1 a_2 \cdot \ldots \cdot a_{n-1}}{b_1 b_2 \cdot \ldots \cdot b_{n-1}} \frac{1}{j_n(j_n - 1)}$$

where $a_i = a_i(j_i)$, $b_i = b_i(j_i)$ $(i = 1, 2, \dots, n-1)$.

Theorem 1. *([14]) The sequence d_n $(n = 1, 2, \dots)$ forms the Markov chain*

$$P(d_1 = j) = \frac{1}{j(j-1)};$$
$$P(d_n = k | d_{n-1} = j) = \frac{h_{n-1}(j)}{k(k-1)}, \quad k > h_{n-1}(j);$$

and 0 otherwise.

Therefore, we get the following properties of cylinders:

1) $\triangle^{ROE}_{j_1 j_2 \dots j_{n-1}} = \bigcup_{i=1}^{\infty} \triangle^{ROE}_{j_1 j_2 \dots j_{n-1} i}$.

2) $sup \triangle^{ROE}_{j_1 j_2 \dots j_n} = inf \triangle^{ROE}_{j_1 j_2 \dots j_{n-1}(j_n - 1)}$.

3) $inf \triangle^{ROE}_{j_1 j_2 \dots j_n} = \frac{1}{j_1} + \frac{a_1}{b_1} \frac{1}{j_2} + \dots + \frac{a_1 a_2 \dots a_n}{b_1 b_2 \dots b_n} \frac{1}{j_n}$,
$sup \triangle^{ROE}_{j_1 j_2 \dots j_n} = \frac{1}{j_1} + \frac{a_1}{b_1} \frac{1}{j_2} + \dots + \frac{a_1 a_2 \dots a_{n-1}}{b_1 b_2 \dots b_{n-1}} \frac{1}{j_n - 1}$.

4) $|\triangle^{ROE}_{j_1 j_2 \dots j_n}| = \frac{a_1 a_2 \dots a_{n-1}}{b_1 b_2 \dots b_{n-1}} \frac{1}{j_n(j_n - 1)}$.

If the first k symbols of ROE are fixed, then $(k+1)$-st symbol of ROE cannot take values $2, 3, \dots, \frac{a_k}{b_k} d_k(d_k - 1)$, $\forall k \in \mathbb{N}$.

Each of the cylinders of ROE can be uniquely rewritten in terms of the difference restricted Oppenheim expansion ($\overline{ROE}$):

$$\alpha_1 = d_1 - 1;$$
$$\alpha_{k+1} = d_{k+1} - \frac{a_k}{b_k} d_k(d_k - 1).$$

Then series (1) can be rewritten as follows:

$$x = \frac{1}{\alpha_1 + 1} + \frac{a_1}{b_1} \frac{1}{\frac{a_1}{b_1} d_1(d_1 - 1) + \alpha_2} + \frac{a_1 a_2}{b_1 b_2} \frac{1}{\frac{a_2}{b_2} d_2(d_2 - 1) + \alpha_3} + \dots =: \Delta^{\overline{ROE}}_{\alpha_1 \alpha_2 \dots \alpha_n \dots}.$$

where $\alpha_k \in \{1, 2, 3, \dots\}$.

Theorem 2. *If there exists a sequence l_k, such that* $\forall x \in [0, 1]$*:*

$$\frac{b_{k-1}}{a_{k-1}} \frac{1}{d_{k-1}(d_{k-1} - 1)} < l_k$$

and series

$$\sum_{k=1}^{\infty} l_k < +\infty,$$

then for any digit i_0 *almost all (with respect to the Lebesgue measure) real numbers* $x \in [0,1]$ *contain symbol* i_0 *only finitely many times in* $\overline{ROE}$.

Proof. Let $N_i(x)$ be a number of symbols "i" in $\overline{ROE}$ of number x. Let us prove that the Lebesgue measure of the set $A_i = \{x : N_i(x) = \infty\}$ is equal to 0 for all $i \in \mathbb{N}$.

Consider the set

$$\bar{\triangle}_i^k = \left\{x : x = \triangle^{\overline{ROE}}_{\alpha_1\alpha_2\ldots\alpha_{k-1}i\alpha_{k+1}\ldots}, \alpha_j \in \mathbb{N}, j \neq k\right\}.$$

From the definition of the set $\bar{\triangle}_i^k$ and properties of cylindrical sets it follows that

$$\bar{\triangle}_i^k = \bigcup_{\alpha_1=1}^{\infty} \cdots \bigcup_{\alpha_{k-1}=1}^{\infty} \triangle^{\overline{ROE}}_{\alpha_1\ldots\alpha_{k-1}i}$$

Let us consider the following ratio:

$$\begin{aligned}
&\frac{|\triangle^{\overline{ROE}}_{\alpha_1\ldots\alpha_{k-1}i}|}{|\triangle^{\overline{ROE}}_{\alpha_1\ldots\alpha_{k-1}}|}\\
&= \frac{|\triangle^{ROE}_{d_1\ldots d_{k-1}(\frac{a_{k-1}}{b_{k-1}}d_{k-1}(d_{k-1}-1)+i)}|}{|\triangle^{ROE}_{d_1\ldots d_{k-1}}|}\\
&= \frac{a_1\ldots a_{k-1}}{b_1\ldots b_{k-1}} \cdot \frac{1}{(\frac{a_{k-1}}{b_{k-1}}d_{k-1}(d_{k-1}-1)+i)(\frac{a_{k-1}}{b_{k-1}}d_{k-1}(d_{k-1}-1)+i-1)}\\
&: \frac{a_1\ldots a_{k-2}}{b_1\ldots b_{k-2}} \cdot \frac{1}{d_{k-1}(d_{k-1}-1)}\\
&\leq \frac{a_{k-1}}{b_{k-1}} \cdot \frac{d_{k-1}(d_{k-1}-1)}{\frac{a_{k-1}}{b_{k-1}}d_{k-1}(d_{k-1}-1)} \cdot \frac{1}{\frac{a_{k-1}}{b_{k-1}}d_{k-1}(d_{k-1}-1)} = \frac{b_{k-1}}{a_{k-1}} \cdot \frac{1}{d_{k-1}(d_{k-1}-1)} < l_k
\end{aligned}$$

Then

$$\lambda\left(\bar{\triangle}_i^k\right) = \sum_{\alpha_1=1}^{\infty} \cdots \sum_{\alpha_k=1}^{\infty} \left|\triangle^{\overline{ROE}}_{\alpha_1\ldots\alpha_{k-1}i}\right| \leq l_k.$$

It is clear, that the set A_i is the upper limit of the sequence of sets $\{\bar{\triangle}_i^k\}$, i.e.,

$$A_i = \limsup_{k\to\infty} \bar{\triangle}_i^k = \bigcap_{m=1}^{\infty}\left(\bigcup_{k=m}^{\infty} \bar{\triangle}_i^k\right).$$

Since

$$\sum_{k=1}^{\infty} \lambda\left(\bar{\triangle}_i^k\right) \leq \sum_{k=1}^{\infty} \frac{b_{k-1}}{a_{k-1}} \frac{1}{d_{k-1}(d_{k-1}-1)} \leq \sum_{k=1}^{\infty} l_k < +\infty,$$

from the Borel–Cantelli Lemma it follows that

$$\lambda(A_i) = 0, \quad \forall i \in N.$$

Therefore,

$$\lambda(\bar{A}_i) = 1, \quad \forall i \in N.$$

Let

$$\bar{A} = \bigcap_{i=1}^{\infty} \bar{A}_i.$$

It is clear that $\lambda(\bar{A}) = 1$, which proves the theorem. □

Example 4. Consider the Sylvester series:

$$d_1 \in \{2, 3, \ldots\},$$
$$d_{k+1} = d_k(d_k - 1) + i, \quad i \in \{1, 2, 3, \ldots\}.$$

If $d_1 = 2$, then the minimal admissible value of d_2 is 3. Therefore

$$\begin{aligned}
d_{k+1} &\geq d_k(d_k - 1) + 1 \geq \big(d_{k-1}(d_{k-1} - 1) + 1\big)\big(d_{k-1}(d_{k-1} - 1) + 1\big) \\
&\geq \big(d_{k-1}(d_{k-1} - 1)\big)^2 + 1 \\
&\geq \Big(\big(d_{k-2}(d_{k-2} - 1) + 1\big)\big(d_{k-2}(d_{k-2} - 1) + 1 - 1\big)\Big)^2 + 1 \\
&\geq \big(d_{k-2}(d_{k-2} - 1)\big)^4 \geq \big(d_{k-3}(d_{k-3} - 1)\big)^{2^3} \geq \cdots \\
&\geq \big(d_{k-(k-2)}(d_{k-(k-2)} - 1)\big)^{2^{k-2}} = \big(d_2(d_2 - 1)\big)^{2^{k-2}} \geq 3 \cdot 2^{2^{k-2}}.
\end{aligned}$$

So for the Sylvester series:

$$\frac{1}{d_{k-1}(d_{k-1} - 1)} < \frac{1}{3 \cdot 2^{2^{k-4}} \cdot (3 \cdot 2^{2^{k-4}})} =: l_k.$$

It is clear that

$$\sum_{k=1}^{\infty} l_k < \infty.$$

Therefore, for λ-almost all $x \in [0, 1]$ their difference Sylvester expansion contain arbitrary digit i only finitely many times.

Example 5. Consider the case where $a_n = d_n$, $b_n = 1$. Then

$$\begin{aligned}
d_{n+1} &\geq d_n \cdot d_n(d_n - 1) + 1 \geq d_n^2 \geq \big(d_{n-1}^2\big)^2 \\
&= d_{n-1}^4 \geq d_{n-2}^8 \geq d_{n-3}^{2^4} \geq \cdots \geq d_{n-(n-1)}^{2^n} = d_1^{2^n} \geq 2^{2^n}.
\end{aligned}$$

So for this case

$$\frac{1}{d_{k-1}(d_{k-1} - 1)} < \frac{1}{2^{2^k}} =: l_k.$$

Then,

$$\sum_{k=1}^{\infty} l_k < \infty.$$

So for λ-almost all $x \in [0, 1]$ the difference expansion contains arbitrary digit i only finitely many times.

3 On singularity of distribution of random variables with independent symbols of the difference restricted Oppenheim expansion

Definition 1. A probability measure μ_ξ of a random variable ξ is said to be singularly continuous (with respect to the Lebesgue measure) if μ_ξ is a continuous probability measure and there exists a set E, such that $\lambda(E) = 0$ and $\mu_\xi(E) = 1$.

Let $x = \Delta^{\overline{ROE}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}$ be $\overline{ROE}$ of real numbers, let $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ be a sequence of independent random variables taking values $1, 2, \ldots, n, \ldots$ with probabilities $p_{1k}, p_{2k}, \ldots, p_{nk}, \ldots$ correspondingly, and let

$$\xi = \Delta^{\overline{ROE}}_{\xi_1\xi_2\ldots\xi_n\ldots}$$

be a random variables with independent $\overline{ROE}$-symbols.

Theorem 3. *Let assumptions of Theorem 2 hold. If there exists a digit i_0 such that $\sum\limits_{k=1}^{\infty} p_{i_0k} = +\infty$, then the probability measure μ_ξ is singular with respect to the Lebesgue measure.*

Proof. Consider sets

$$\bar{\Delta}^n_{i_0} = \{x : \alpha_n(x) = i_0\}$$

and

$$A_{i_0} = \{x : N_{i_0}(x) = +\infty\}.$$

It is clear, that $A_{i_0} = \overline{\lim\limits_{n\to\infty}}\, \bar{\Delta}^n_{i_0}$.

From the definition of $\bar{\Delta}^n_{i_0}$ it follows that $\mu_\xi(\bar{\Delta}^n_{i_0}) = p_{i_0n}$.

Since the random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are independent, we conclude that

$$\begin{aligned}
&\mu_\xi\big(\bar{\Delta}^{k_1}_{i_0} \cap \bar{\Delta}^{k_2}_{i_0} \cap \cdots \cap \bar{\Delta}^{k_s}_{i_0}\big)\\
&\quad = \mu_\xi\big(\{x : \alpha_{k_1}(x) = i_0, \alpha_{k_2}(x) = i_0, \ldots, \alpha_{k_s}(x) = i_0\}\big)\\
&\quad = \mu_\xi\big(\{x : \alpha_{k_1}(x) = i_0\}\big) \cdot \mu_\xi\big(\{x : \alpha_{k_2}(x) = i_0\}\big) \cdot \ldots \cdot \mu_\xi\big(\{x : \alpha_{k_s}(x) = i_0\}\big)\\
&\quad = p_{i_0k_1} \cdot p_{i_0k_2} \cdot \ldots \cdot p_{i_0k_s}.
\end{aligned}$$

So, events $\bar{\Delta}^1_{i_0}, \bar{\Delta}^2_{i_0}, \ldots, \bar{\Delta}^n_{i_0}, \ldots$ are independent with respect to measure μ_ξ.

Since $\sum\limits_{k=1}^{\infty} p_{i_0k} = +\infty$ and $\{\bar{\Delta}^n_{i_0}\}$ is a sequence of independent events, from the Borel–Cantelli Lemma it follows that

$$\mu_\xi(A_{i_0}) = 1.$$

Let λ be the Lebesgue measure. Events $\bar{\Delta}^1_{i_0}, \bar{\Delta}^2_{i_0}, \ldots, \bar{\Delta}^n_{i_0}, \ldots$, in general, are not independent w.r.t. the Lebesgue measure. We estimate the Lebesgue measure of the set $\bar{\Delta}^n_{i_0}$:

$$\lambda\big(\bar{\Delta}^n_{i_0}\big) = \lambda\big(\{x : \alpha_n(x) = i_0\}\big)$$

$$= \sum_{\alpha_1(x)=1}^{\infty} \sum_{\alpha_2(x)=1}^{\infty} \cdots \sum_{\alpha_{n-1}=1}^{\infty} \left|\Delta^{\overline{ROE}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_{n-1}(x)i_0}\right|$$

$$= \sum_{\alpha_1(x)=1}^{\infty} \sum_{\alpha_2(x)=1}^{\infty} \cdots \sum_{\alpha_{n-1}=1}^{\infty} \frac{\left|\Delta^{\overline{ROE}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_{n-1}(x)i_0}\right|}{\left|\Delta^{\overline{ROE}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_{n-1}(x)}\right|} \cdot \left|\Delta^{\overline{ROE}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_{n-1}(x)}\right|$$

$$\leq l_n(i_0) \cdot \sum_{\alpha_1(x)=1}^{\infty} \sum_{\alpha_2(x)=1}^{\infty} \cdots \sum_{\alpha_{n-1}=1}^{\infty} \left|\Delta^{\overline{ROE}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_{n-1}(x)}\right| = l_n \cdot 1,$$

where l_n are defined in Theorem 2. Therefore,

$$\sum_{n=1}^{\infty} \lambda\left(\bar{\Delta}^n_{i_0}\right) \leq \sum_{n=1}^{\infty} l_n < +\infty.$$

So by the Borel–Cantelli Lemma, $\lambda(A_{i_0}) = 0$, i.e. for λ-almost all $x \in [0, 1]$ their $\overline{ROE}$ contains arbitrary digit i only finitely many times.

Hence $\lambda(A_{i_0}) = 0$, and $\mu_\xi(A_{i_0}) = 1$. So, probability measure μ_ξ is singular with respect to the Lebesgue measure □

Theorem 4. *Let assumptions of Theorem 2 hold. If ξ_k are independent and identically distributed random variables, then the probability measure μ_ξ is singular with respect to the Lebesgue measure.*

Proof. If $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are independent and identically distributed random variables, then $p_{ik} = p_i$.

Since $\sum_{i=1}^{\infty} p_i = 1$, it is clear that there exists a number i_0 such, that: $p_{i_0} > 0$. Therefore

$$\sum_{k=1}^{\infty} p_{i_0 k} = +\infty,$$

and the singularity of μ_ξ follows directly from Theorem 3. □

Corollary 1. *Let*

$$x = \Delta^{\overline{S}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots}$$

be the difference version of the Sylvester expansion ($\overline{S}$-expansion) and let

$$\xi = \Delta^{\overline{S}}_{\xi_1\xi_2\ldots\xi_n\ldots}$$

be the random variable with independent symbols of $\overline{S}$-expansion.

If there exists a digit i_0 such that $\sum_{k=1}^{\infty} p_{i_0 k} = +\infty$, then the probability measure μ_ξ is singular with respect to the Lebesgue measure.

In particular, the distribution of the random variable with independent identically distributed symbols of $\overline{S}$-expansion is singular w.r.t. the Lebesgue measure.

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